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ul. Powstańców Wielkopolskich 16, 61-895 Poznań, Poland
phone +48 61 854 31 54, +48 61 854 31 55, fax +48 61 854 31 59
www.wydawnictwo-ue.pl, e-mail: wydawnictwo@ue.poznan.pl
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Bipolar growth model with investment flows[1]

***Katarzyna Filipowicz*[2], *Tomasz Misiak*[3], *Tomasz Tokarski*[2]**

Abstract: The aim of the present study is to design a bipolar model of economic growth with investment flows between two types of economies (conventionally referred to as relatively rich economies and relatively poor economies). Therefore in the following considerations it is assumed that the process of capital accumulation depends on investments undertaken in the economy. At the same time the Solow growth model takes into account only investments financed by domestic savings, whereas in the bipolar growth model also the investment flows between rich and poor economies are considered. It is assumed that both relatively rich economies are investing in the relatively poor economies and the poor economies make investments in the rich economies.

The paper analyses the long-term equilibrium of the growth model, both in terms of existence of steady states of the system of differential equations and in terms of the stability of a non-trivial steady state. What is more economic characteristics of the point of the long-term equilibrium of the model are examined, model parameters are calibrated and growth paths of basic macroeconomic variables in selected variants of numerical simulations are presented.

Keywords: economic growth, investment flows, convergence, numerical simulations.

JEL codes: O4, O410, O470, C020.

1. Introduction and literature review

The theory of economic growth is one of the most interesting research areas of contemporary macroeconomics. Economic growth is said to be an antidote to contemporary problems of developing economies despite the fact that 50 years ago Evsey Domar claimed that "economic growth occupied a strange place in the theory of economics: it was always perceived in the neighbourhood, around, but it was rarely invited inside" [Domar 1962: 51].

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[2] Jagiellonian University, Department of Mathematical Economics, ul. S. Łojasiewicza 4, 30-348 Kraków, Poland.

[3] Rzeszów University of Technology, Department of Economics, al. Powstańców Warszawy 8, 35-959 Rzeszów, Poland; corresponding author: tmisiak@prz.edu.pl.

It transpires that the models of economic growth concentrate their attention on the inclusion of the processes of technological progress and the accumulation of broadly-understood capital in the long run, in order to explain the causes of the commonly occurring diversification of levels as well as the rates of economic growth. Neoclassical models as well as the models of new growth theory omit spatial interactions. The inclusion of spatial interaction (location) and accumulation processes (growth) belong to the interesting and simultaneously most difficult research areas the contemporary results of which (theoretical as well as empirical) are not satisfactory [Combes, Mayer, and Thisse 2008]. The omission of interactions and the aspatial nature of the economic mainstream growth theory until the appearance of the models of *new economic geography* (NEG) should be treated as a certain weakness of this theory. It is also pointed out by Malaga [2011] who considers that "the space in the theory of economic growth is usually treated in a trivial way, separately from the achievements of the economic spatial analysis. The spatial aspect appears implicit in relation to the comparative analysis of economic growth or development processes in different countries or group of countries [...]. It does not change the fact that from the viewpoint of economic spatial analysis the theory of economic growth and development is of an aspatial character, and the mechanisms and processes of economic growth usually fail to have spatial location." However the inclusion of space and reciprocal interactions in theoretical considerations, thus the endogenisation of localisational choices requires the exceeding of the simplified frameworks of contemporary neoclassical models. In the basic models of economic growth it is usually assumed that a particular country or region is an island, the main growth power of which are the internal potential and investments are determined exclusively by domestic savings. This assumption appears to excessively simplify the reality and particularly in the time of the ongoing globalisation processes and growing economic integration at national and regional levels with free investment, people, products or technologies flow. Thus it transpires that space (location) combined with the accumulation processes are of crucial significance and in the contemporary growth models location was de facto insignificant.

Aiming to present a broader theoretical context for the role of space and spatial interactions in the processes of economic growth from the viewpoint of the achievements of contemporary economics one should analyse the output of the two primary theoretical streams:

- the theory of economic growth;
- the theory of spatial interactions (location) including new economic geography (NEG).

The starting point for the considerations on the theory of economic growth is most frequently Solow's analysis from 1957. In spite of the fact that according to Solow: "the theory of growth has not started with my articles from 1956 and 1957 and will certainly not end with them, it may have started with Wealth Nations, but probably even Adam Smith had his predecessors" [Solow 1957], in the literature of the subject his analyses are considered ground -breaking in the scope of the calculations of the influence of capital accumulation and technological progress on economic growth rate. These analyses are also known as the Solow residual equation and have been widely applied not only in the models of the real business cycle but also in estimations of total factor productivity. One of the more interesting conclusions arising from the Solow model is the effect of convergence of development levels in the long-run into a common one (β – absolute convergence) or into individual ones, because of the structural aspects of economies (β – conditional convergence). What is also interesting is the club convergence hypothesis proposed by Baumol [1986] which assumes the presence of a catch-up effect provided that the economies are similar in terms of their basic characteristics. The majority of the subsequent research into the convergence processes has been derived from the research by Baumol [1986], Barro and Sala-i-Martin [1992] and Mankiw, Romer, and Weil [1992].

The Mankiw, D. Romer, and Weil (MRW) model [1992] constitutes an extension of the Solow model taking into account human capital accumulation. The Solow and MRW models analyse the influence of the level and structure of investment rates on the location as well as the slope of the long-run path of economic growth. Moreover constant returns to scale lead to a certain stability of economic growth rates at the level determined by the rate of exogenous technological progress.

The next step in the development of the theory of economic growth was the attempt to endogenise economic growth within the scope of the *new growth theory* (NGT). Presently the first and the second generation of NGT models is distinguished The first generation is represented by the Romer [1986], Lucas [1988] and Barro [1990] models as well as the model of AK Rebelo type [1991]. In these models the economists deviated from the neoclassical assumption of the decreasing marginal productivity of physical capital, allowing for the presence of a constant or growing marginal product from the broadly understood capital. Thus external benefits ensuing from human capital accumulation and the effect of spreading knowledge were achieved inter alia, owing to the reference to the concept of *learning by doing*. The second generation of NGT models focused on explaining technological progress leading to growth endogenisation. It required the introduction into the analysis of the research and development sector which generates knowledge or innovations. The inclusion of R&D activity in the growth models was initiated by Romer [1987, 1990], Aghion and Howitt [1992] as well as in the Grossman and Helpman models [1990, 1991].

Further development of economic growth modelling was orientated towards the inclusion of institutional conditions [Acemoglu, Johnson, and Robinson

2001] and exogenic geographic conditions [cf. e.g. Gallup and Sachs 1999 or Rodrik 2002].[4]

The appearance of the first models of new economic geography (NEG) has restored the issue of space in the major stream of the theory of economics. The person that is considered to have established NEG is Krugman [1991] who in his paper draws attention to the significance of localisation in shaping economic processes. In Krugman's analyses Myrdal's agglomeration theory and cumulative causality [Mydral 1957] have been combined in a concept of circular causality.[5] The Krugman model with a core-periphery system has been created on the basis of the model of the new trade theory by Krugman [1980], which has been an extention of interregional mobility of production factors. This model, in contrast to economic growth models, is based on the concept of monopolistic competition by Dixit-Stiglitz [1977], hence it is frequently called the DSK model (Dixit-Stiglitz-Krugman). One of the key features of NEG models is the issue of endogenisation of localisation decisions in space. It is households and enterprises that take conscious decisions on localisation taking into account the maximisation of the function of total utility or profit. The level of spatial concentration or dispersion of economic activity depends on centripetal forces (pro-agglomerate) as well as the centrifugal forces (pro-disperse). NEG models based on the concept of monopolistic competition allow for the presence of internal as well as external economies of scale; however the external economies of scale become a natural pro-agglomerate force. The NEG model in the DSK formula is based on two types of regions (e.g. rich north – poor south), two economic sectors (agriculture and processing industry), two production factors (capital and work), whereas one of the factors is mobile and the other immobile.

However it transpires that the majority of NEG models are of a static character. They explain the evolution of the placement of activities in space but omit the problem of accumulation. Thus there arises a basic question as to how the change of location of an economic activity (e.g., through investment flow) influences economic growth of particular economies as well as the convergence process. The understanding of this type of interaction requires the synthesis of the NEG and the economic growth models. One of the first attempts at such synthesis are the papers by Martin and Ottaviano [1999] or Baldwin and Forslid [2000] which constitute a generation of dynamic NEG models. These models emphasise the significance of human capital and knowledge accumulation, however these effects decrease proportionally to the distance, thus they

[4] One should notice that primary geographic conditions (physical geography) were actually of marginal significance. What is of crucial importance is the second nature of the geographic, which is connected to the present dislocation of people and economic potential, including the main industrial and metropolitan areas.

[5] A chain of cause and effect relationships of various occurrences creating a circle of feedbacks.

are positioned in space. Hence the factors that determine the localisation of economic activity in dynamic NEG models become the factors that in endogenous growth models determine the growth.

Baldwin, et al. [2003] considers that in a static depiction of NEG models mobile resources of the factors of production are constant. Thus if there is a move within regions or countries, the accumulation process which could change the resource amount of the available factors does not occur. In dynamic NEG models resources can change in time which principally differentiates both approaches.

Economic growth as well as the issue of activity localisation (e.g. related to direct investment flow internationally) is of an endogenous character. Between these processes there certain interactions occur which are only taken into account to a limited degree in contemporary theoretical models. In the present paper the authors attempt to build a model of economic growth that takes into consideration the spatial aspect and the mutual interactions occurring between two economies (the relatively poor and the relatively rich).

2. The assumptions of the model[6]

In the following considerations there the following assumptions regarding the functioning of two types of economies are adopted:[7]

1. A production process in i economy (for i = 1, 2) is described with a function of labour productivity of Cobb-Douglas type (1928) given by the formula:

$$\forall i, j = 1{,}2 \ \wedge \ j \neq i \quad y_i(t) = \left(k_j(t)\right)^{\beta} \left(k_i(t)\right)^{\alpha}, \tag{1}$$

where:

$\alpha, \beta, (\alpha + \beta) \in (0; 1)$ and $\alpha > \beta$.

Parameter α denotes the flexibility of labour productivity y_i (production Y_i) in i economy in relation to capital-labour ratio k_i (capital K_i) in this economy or (on the basis of marginal productivity theory of distribution by Clark) capital-product ratio. Parameter β denotes the flexibility of total factor

[6] The preliminary Polish version of the model has been published in: [Filipowicz and Tokarski 2015].

[7] All macroeconomic variables included in points 3–5 are assumed to be differentiable functions of time $t \geq 0$. The formula $x(t)$ denotes that the value of variable x in t time, and $\dot{x}(t) = dx/dt$ – a derivate of variable x after t time, e.g., (in economic terms) the increase of the value of this variable in t time.

productivity[8] in i-economy in relation to capital-labour ratio in j-economy (for $j \neq i$).[9]

The influence of capital-labour ratio in j economy on labour productivity in i economy (for $i \neq j$) can be explained in three ways. Firstly as in the gravity model of economic growth [proposed by Mroczek, Tokarski and Trojak 2014], it can ensue from the gravity effect. Secondly, the influence of k_j on y_i, where $k_j > k_i$, can result from the fact that poor economies (which by means of imitation absorb new technological solutions) use a high capital-labour ratio in relatively rich economies. Thirdly, the effectiveness of the function of relatively poor economies is positively influenced by a better developed infrastructure (e.g. transport) in richer economies, however the effectiveness of the function of rich economies is negatively influenced by the less developed infrastructure of poor economies.

From the function of labour productivity it ensues also that:

$$\frac{y_1(t)}{y_2(t)} = \left(\frac{k_1(t)}{k_2(t)}\right)^{\alpha-\beta}. \tag{2}$$

The relationships between the quotients of capital-labour ratios k_1/k_2 and labour productivity y_1/y_2 can be illustrated as in Figure 1. It ensues from formula (2) and Figure 1 that if the divergence of the capital-labour ratio exists (that is $k_1/k_2 \neq 1$), this divergence must also translate into the divergence on the part of labour productivity (therefore also $y_1/y_2 \neq 1$). Moreover owing to the fact

[8] In Cobb-Douglas production function given by formula:

$$Y = AK^{\alpha}L^{1-\alpha},$$

where Y denotes the stream of the generated product, K – physical capital input, L – labour input, and $\alpha \in (0; 1)$; total factor productivity A (subscripts i referring to the following economies are omitted here) can be identified with product Y generated from unit capital input K and work input L, or:

$$A = \frac{Y}{K^{\alpha}L^{1-\alpha}} = \left(\frac{Y}{K}\right)^{\alpha}\left(\frac{Y}{L}\right)^{1-\alpha} = p^{\alpha}y^{1-\alpha},$$

which – the total factor productivity A is a geometric weighted average of capital productivity ($p = Y/K$) and labour productivity ($y = Y/L$), where the role of weights is played by production-capital (α) and production-labour ratios ($1 - \alpha$) in the product generated in the economy.

[9] If the functions of labour productivity (1) are multiplied by $L_i > 0$ (for $i = 1, 2$), there are obtained production functions given by the formula:

$$\forall i, j = 1,2 \wedge j \neq i \;\; Y_i = k_j^{\beta}K_i^{\alpha}L_i^{1-\alpha},$$

from which it can be inferred that these functions are homogeneous of order 1 in relation to K_I and L_i and homogeneous of order $1 + \beta > 1$ in relation to K_I, L_i and k_j. Thus, production functions that are derived from the function of labour productivity (1) are characterised by constant return-to-scale in relation to K_I and L_i and increasing return-to-scale in relation to K_I, L_i and k_j.

that the flexibility of y_1/y_2 in relation to k_1/k_2 (equal to $\alpha - \beta$) is smaller than 1, it ensues from the above assumptions that the diversity of labour productivity in the model of economic growth analysed is smaller than the diversification of capital-labour ratio.

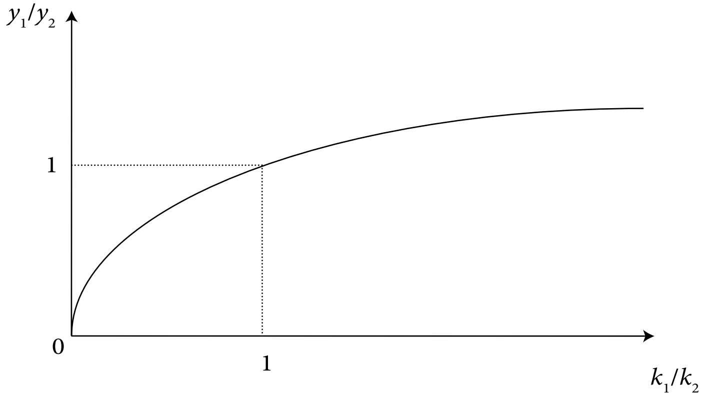

Figure 1. Relationships between the quotients of capital-labour k_1/k_2 and labour productivity y_1/y_2

2. The increase in capital input $\dot{K}_i$ (for $i = 1, 2$) in each of the analysed types of economies is described by the following differential equations:

$$\forall i, j = 1,2 \ \wedge \ j \neq i \quad \dot{K}_i(t) = s_{ii}Y_i(t) + s_{ij}Y_j(t) - \delta_i K_i(t), \tag{3}$$

where:

$\forall i, j = 1,2 \ \ s_{ij} \in (0;1),\ s_{11} + s_{21} \in (0;1),\ s_{22} + s_{12} \in (0;1),\ s_{11} \geq s_{21},\ s_{22} \geq s_{12},$

$\forall i = 1,2 \ \ \delta_i \in (0;1),$

$\forall i = 1,2 \ \ Y_i = y_i L_i$, where L_i denotes the number of the workers in the economy i, and Y_i – the production volume in this economy.

Denotations s_{ii} and s_{ij} denote respectively the size of investments financed by economy i (s_{ii}) and identified in the economy i or economy j (s_{ij}). Therefore the values s_{ij} (for $j \neq i$) can be interpreted as foreign investments from economy j to economy i. Furthermore δ_i (for $i = 1, 2$) denotes the rates of capital depreciation in the economies of i type.

3. The number of the workers in both types of economies grows according to the same growth rate $n > 0$, which means that:

$$L(t) = L_o e^{nt}, \tag{4}$$

where:

$L_0 > 0$ is the aggregate number of the workers (in both types of economies) in time $t = 0$.

4. Economy 1 absorbs the percentage of the aggregate number of the workers (in both types of economies) equal to $\omega \in (0; 1)$, whereas economy 2 absorbs the percentage of the workers amounting to $1 - \omega$. Hence it can be concluded that:

$$L_1(t) = \omega L_0 e^{nt} \;\Rightarrow\; \frac{\dot{L}_1(t)}{L_1(t)} = n \tag{5}$$

and:

$$L_2(t) = (1-\omega) L_0 e^{nt} \;\Rightarrow\; \frac{\dot{L}_2(t)}{L_2(t)} = n. \tag{6}$$

3. The equilibrium of the model

Due to the fact that for each $i = 1, 2$ the capital K_i can be expressed as $k_i L_i$:

$$\forall i = 1,2 \quad \dot{k}_i(t) = \frac{\dot{K}_i(t) L_i(t) - K_i(t)\dot{L}_i(t)}{\left(L_i(t)\right)^2} = \frac{\dot{K}_i(t)}{L_i(t)} - \frac{\dot{L}_i(t)}{L_i(t)} k_i(t),$$

which – together with the assumptions (5–6) – gives:

$$\forall i = 1,2 \quad \dot{k}_i(t) = \frac{\dot{K}_i(t)}{L_i(t)} - n k_i(t). \tag{7}$$

By introducing relationships (3) to equations (7) one obtains:

$$\forall i, j = 1,2 \wedge j \neq i \quad \dot{k}_i(t) = s_{ii} y_i(t) + s_{ij} \frac{L_j(t)}{L_i(t)} y_j(t) - \mu_i k_i(t), \tag{8}$$

where:

$\forall i = 1,2 \;\; \mu_i = \delta_i + n > 0$ denotes the ratio of reduction in capital per worker.
After inserting equations (5–6) into relationships (8) there is achieved:

$$\dot{k}_1(t) = s_{11} y_1(t) + s_{12} \frac{1-\omega}{\omega} y_2(t) - \mu_1 k_1(t)$$

and:

$$\dot{k}_2(t) = s_{22} y_2(t) + s_{21} \frac{\omega}{1-\omega} y_1(t) - \mu_2 k_2(t),$$

and hence, after taking into account the function of labour productivity (1), the following system of non-linear differential equations is obtained:

$$\begin{cases} \dot{k}_1(t) = s_{11}\left(k_1(t)\right)^{\alpha}\left(k_2(t)\right)^{\beta} + s_{12}\dfrac{1-\omega}{\omega}\left(k_1(t)\right)^{\beta}\left(k_2(t)\right)^{\alpha} - \mu_1 k_1(t), \\ \dot{k}_2(t) = s_{22}\left(k_2(t)\right)^{\alpha}\left(k_1(t)\right)^{\beta} + s_{21}\dfrac{\omega}{1-\omega}\left(k_2(t)\right)^{\beta}\left(k_1(t)\right)^{\alpha} - \mu_2 k_2(t). \end{cases} \tag{9}$$

The system of differential equations (9) has in its phase space $P = [0; +\infty)^2$ two steady states: trivial point (0; 0) and a certain non-trivial point $k^{\star} = (k_1^{\star}; k_2^{\star}) \in (0; +\infty)^2$, which will be soon found.[10]

Non-trivial steady states k^* solves the system of equations:

$$\begin{cases} s_{11}\left(k_1^{\star}\right)^{\alpha}\left(k_2^{\star}\right)^{\beta} + s_{12}\dfrac{1-\omega}{\omega}\left(k_1^{\star}\right)^{\beta}\left(k_2^{\star}\right)^{\alpha} = \mu_1 k_1^{\star}, \\ s_{22}\left(k_1^{\star}\right)^{\beta}\left(k_2^{\star}\right)^{\alpha} + s_{21}\dfrac{\omega}{1-\omega}\left(k_1^{\star}\right)^{\alpha}\left(k_2^{\star}\right)^{\beta} = \mu_2 k_2^{\star}. \end{cases} \tag{10}$$

Owing to the fact that in non-trivial steady states $k^{\star}$: $\dot{k}_1 = \dot{k}_2 = 0$ and $k^{\star} \in \left(0; +\infty\right)^2$, the quotient $k_1^{\star}/k_2^{\star}$ is a certain positive real number κ. Thus, $k_1^{\star} = \kappa k_2^{\star}$, which causes that the system of equations (10) can be expressed as follows:

$$\begin{cases} \left(u_1\kappa^{\alpha} + v_1\kappa^{\beta}\right)\left(k_2^{\star}\right)^{\alpha+\beta} = \kappa k_2^{\star}, \\ \left(u_2\kappa^{\beta} + v_2\kappa^{\alpha}\right)\left(k_2^{\star}\right)^{\alpha+\beta} = k_2^{\star}, \end{cases} \tag{11}$$

where:

$$\forall i = 1, 2 \;\; u_i = \frac{s_{ii}}{\mu_i} > 0,$$

$$v_1 = \frac{s_{12}(1-\omega)}{\mu_1\omega} > 0,$$

$$v_2 = \frac{s_{21}\omega}{\mu_2(1-\omega)} > 0.$$

By dividing the first equation of the system (11) by the second one, there is obtained:

$$\kappa = \frac{u_1\kappa^{\alpha} + v_1\kappa^{\beta}}{u_2\kappa^{\beta} + v_2\kappa^{\alpha}},$$

[10] Trivial point (uninteresting from an economic or mathematical viewpoint) will be omitted in the following analyses.

which leads to the relationship:

$$\phi(\kappa) = u_2\kappa^{\beta+1} + v_2\kappa^{\alpha+1} - u_1\kappa^{\alpha} - v_1\kappa^{\beta} = 0. \tag{12}$$

The function $\phi(\kappa)$ is characterised by the following features:

(i) $\phi(0) = 0$,

(ii) $\lim_{\kappa\to+\infty}\phi(\kappa) = +\infty$,

(iii) $\phi'(\kappa) = (\beta+1)u_2\kappa^{\beta} + (\alpha+1)v_2\kappa^{\alpha} - \alpha u_1\kappa^{\alpha-1} - \beta v_1\kappa^{\beta-1}$ for each $\kappa > 0$,

(iv) $\lim_{\kappa\to0^+}\phi'(\kappa) = -\infty$ and $\lim_{\kappa\to0^+}\phi'(\kappa) = +\infty$,

(ϖ) $\forall\kappa > 0\ \ \phi''(\kappa) = \beta(\beta+1)u_2\kappa^{\beta-1} + \alpha(\alpha+1)v_2\kappa^{\alpha-1} + (1-\alpha)\alpha u_1\kappa^{\alpha-2} +$
$+ (1-\beta)\beta v_1\kappa^{\beta-2} > 0$,

which together with the characteristics (iv) guarantees that there is exactly one $\overline{\kappa}$, which for each $\kappa\in(0;\overline{\kappa})$: $\phi'(\kappa) < 0$, and for $\kappa\in(\overline{\kappa};+\infty)$: $\phi'(\kappa) > 0$. Thus in the range $\kappa\in(0;\overline{\kappa})$ the function $\phi(\kappa)$ decreases and in the range $(\overline{\kappa};+\infty)$ – the function increases. Furthermore owing to the fact that $\phi(\overline{\kappa}) < 0$ and $\lim_{\kappa\to+\infty}\phi(\kappa) = +\infty$, there is also exactly one $\kappa^* > \overline{\kappa}$ that solves the equation (12).

Due to the fact that there is exactly one $\kappa^* > 0$ that solves equation (12) from the first equation of the system of equations (11) a conclusion can be drawn that there is also exactly one capital-labour ratio $k_2^* > 0$ that solves this system of equations. It denotes that the system of differential equations (9) has exactly one non-trivial steady states $k^* \in (0; +\infty)^2$.

The Jacobian matrix of the system of differential equations (9) determines the relation:

$$J = \begin{bmatrix} \alpha s_{11}k_1^{\alpha-1}k_2^{\beta} + \beta s_{12}\dfrac{1-\omega}{\omega}k_1^{\beta-1}k_2^{\alpha} - \mu_1 & \dfrac{\beta s_{11}k_1^{\alpha}k_2^{\beta} + \alpha s_{12}\dfrac{1-\omega}{\omega}k_1^{\beta}k_2^{\alpha}}{k_2} \\ \dfrac{\beta s_{22}k_1^{\beta}k_2^{\alpha} + \alpha s_{21}\dfrac{\omega}{1-\omega}k_1^{\alpha}k_2^{\beta}}{k_1} & \alpha s_{22}k_1^{\beta}k_2^{\alpha-1} + \beta s_{21}\dfrac{\omega}{1-\omega}k_1^{\alpha}k_2^{\beta-1} - \mu_2 \end{bmatrix}. \tag{13}$$

Due to the fact that in steady states k^*, according to the equations (10), there are the relationships:

$$\mu_1 = s_{11}\left(k_1^*\right)^{\alpha-1}\left(k_2^*\right)^{\beta} + s_{12}\frac{1-\omega}{\omega}\left(k_1^*\right)^{\beta-1}\left(k_2^*\right)^{\alpha}$$

and

$$\mu_2 = s_{22}\left(k_1^*\right)^{\beta}\left(k_2^*\right)^{\alpha-1} + s_{21}\frac{\omega}{1-\omega}\left(k_1^*\right)^{\alpha}\left(k_2^*\right)^{\beta-1}.$$

At this point the Jacobian matrix (13) can be expressed as follows:

$$J^* = \begin{bmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{bmatrix}, \tag{14}$$

where:

$$j_{11} = -\frac{(1-\alpha)s_{11}\left(k_1^*\right)^{\alpha}\left(k_2^*\right)^{\beta} + (1-\beta)s_{12}\dfrac{1-\omega}{\omega}\left(k_1^*\right)^{\beta}\left(k_2^*\right)^{\alpha}}{k_1^*} < 0,$$

$$j_{22} = -\frac{(1-\alpha)s_{22}\left(k_1^*\right)^{\beta}\left(k_2^*\right)^{\alpha} + (1-\beta)s_{21}\dfrac{\omega}{1-\omega}\left(k_1^*\right)^{\alpha}\left(k_2^*\right)^{\beta}}{k_2^*} < 0,$$

$$j_{12} = \frac{\beta s_{11}\left(k_1^*\right)^{\alpha}\left(k_2^*\right)^{\beta} + \alpha s_{12}\dfrac{1-\omega}{\omega}\left(k_1^*\right)^{\beta}\left(k_2^*\right)^{\alpha}}{k_2^*} > 0,$$

$$j_{21} = \frac{\beta s_{22}\left(k_1^*\right)^{\beta}\left(k_2^*\right)^{\alpha} + \alpha s_{21}\dfrac{\omega}{1-\omega}\left(k_1^*\right)^{\alpha}\left(k_2^*\right)^{\beta}}{k_1^*} > 0.$$

The eigenvalues of the Jacobian matrix (14) are the elements of the following equation:

$$\lambda^2 - trJ^*\lambda + \det J^* = 0. \tag{15}$$

The elements of equation (15) are real numbers since:

$$\Delta = (trJ^*)\ -4\det J^* = (j_{11} + j_{22})^2 - 4(j_{11}j_{22} - j_{12}j_{21}) =$$

$$= (j_{11} - j_{22})^2 + 4j_{12}j_{21} \geq 4j_{12}j_{21} > 0.$$

It ensues from Vieta's formulas and equation (15) that the sum $\lambda_1 + \lambda_2$ and the product $\lambda_1\lambda_2$ of the eigenvalues of the the Jacobian matrix J^* are determined by the relationships:

$$\lambda_1 + \lambda_2 = trJ^*$$

and:

$$\lambda_1 \lambda_2 = \det J^*.$$

Since $trJ^* = j_{11} + j_{22} < 0$, the sum of the eigenvalues is a negative real number.
Moreover the following relationships occur:

$$j_{11}j_{22} = \frac{\left((1-\alpha)^2 s_{11}s_{22} + (1-\beta)^2 s_{12}s_{21}\right)\left(k_1^* k_2^*\right)^{\alpha+\beta}}{k_1^* k_2^*} + \\ + \frac{(1-\alpha)(1-\beta)\left(s_{11}s_{21}\frac{\omega}{1-\omega}\left(k_1^*\right)^{2\alpha}\left(k_2^*\right)^{2\beta} + s_{22}s_{12}\frac{1-\omega}{\omega}\left(k_1^*\right)^{2\beta}\left(k_2^*\right)^{2\alpha}\right)}{k_1^* k_2^*}$$

and:

$$j_{12}j_{21} = \\ = \frac{\left(\beta^2 s_{11}s_{22} + \alpha^2 s_{12}s_{21}\right)\left(k_1^* k_2^*\right)^{\alpha+\beta} + \alpha\beta\left(s_{11}s_{21}\frac{\omega}{1-\omega}\left(k_1^*\right)^{2\alpha}\left(k_2^*\right)^{2\beta} + s_{22}s_{12}\frac{1-\omega}{\omega}\left(k_1^*\right)^{2\beta}\left(k_2^*\right)^{2\alpha}\right)}{k_1^* k_2^*}.$$

It results from $\det J^* = j_{11}j_{22} - j_{12}j_{21}$ that:

$$\det J^* = \frac{\left(\left((1-\alpha)^2 - \beta^2\right)s_{11}s_{22} + \left((1-\beta)^2 - \alpha^2\right)s_{12}s_{21}\right)\left(k_1^* k_2^*\right)^{\alpha+\beta}}{k_1^* k_2^*} + \\ + \frac{(1-\alpha-\beta)\left(s_{11}s_{21}\frac{\omega}{1-\omega}\left(k_1^*\right)^{2\alpha}\left(k_2^*\right)^{2\beta} + s_{22}s_{12}\frac{1-\omega}{\omega}\left(k_1^*\right)^{2\beta}\left(k_2^*\right)^{2\alpha}\right)}{k_1^* k_2^*} = \\ = \frac{\left((1-\alpha-\beta)(1-\alpha+\beta)s_{11}s_{22} + (1-\alpha-\beta)(1+\alpha-\beta)s_{12}s_{21}\right)\left(k_1^* k_2^*\right)^{\alpha+\beta}}{k_1^* k_2^*} + \\ + \frac{(1-\alpha-\beta)\left(s_{11}s_{21}\frac{\omega}{1-\omega}\left(k_1^*\right)^{2\alpha}\left(k_2^*\right)^{2\beta} + s_{22}s_{12}\frac{1-\omega}{\omega}\left(k_1^*\right)^{2\beta}\left(k_2^*\right)^{2\alpha}\right)}{k_1^* k_2^*} > 0,$$

which denotes that $\det J^* > 0$. Hence it can be concluded that both of the eigenvalues of the Jacobian matrix J^* are negative real numbers and hence (on the basis of Grobman-Hartman theorem, cf. Ombach [1999, theorem 6.2.1]) it ensues that steady states k^* of the system of differential equations (9) is an asymptotically stable point. Therefore this point is a point of a long-run equilibrium of the economic growth model discussed in the paper.

4. Economic characteristics of the model

It can be concluded from the above considerations that the quotient κ^*, which is a relationship of capital-labour ratios k_1^*/k_2^*, is a certain implicit function u_1, u_2, v_1 and v_2. Thus:

$$\kappa^* = \kappa^*(u_1, u_2, v_1, v_2). \tag{16}$$

Furthermore, since from the substitution made in point 4 it ensues that for each $i = 1, 2$ $u_i = u_i(s_{ii}, \mu_i)$ and $v_1 = v_1(s_{12}, \mu_1, \omega)$ and $v_2 = v_2(s_{21}, \mu_2, \omega)$, from this and from equation (16) it can be concluded that:

$$\kappa^* = \kappa^*\big(u_1(s_{11}, \mu_1), u_2(s_{22}, \mu_2), v_1(s_{12}, \mu_1, \omega), v_2(s_{21}, \mu_2, \omega)\big). \tag{17}$$

By applying the formulas for implicit function derivatives it can be depicted that the partial derivatives of function (17) determined by the relationships:[11]

$$\frac{\partial \kappa^*}{\partial s_{11}} = -\frac{\frac{\partial \varphi}{\partial u_1}\frac{\partial u_1}{\partial s_{11}}}{\partial \phi / \partial \kappa} = \frac{(\kappa^*)^{\alpha}}{\mu_1 \partial \phi / \partial \kappa} > 0, \tag{18}$$

$$\frac{\partial \kappa^*}{\partial s_{22}} = -\frac{\frac{\partial \phi}{\partial u_2}\frac{\partial u_2}{\partial s_{22}}}{\partial \phi / \partial \kappa} = -\frac{(\kappa^*)^{\beta+1}}{\mu_2 \partial \phi / \partial \kappa} < 0, \tag{19}$$

$$\frac{\partial \kappa^*}{\partial s_{12}} = -\frac{\frac{\partial \phi}{\partial v_1}\frac{\partial v_1}{\partial s_{12}}}{\partial \phi / \partial \kappa} = \frac{(1-\omega)(\kappa^*)^{\beta}}{\mu_1 \omega \partial \phi / \partial \kappa} > 0, \tag{20}$$

$$\frac{\partial \kappa^*}{\partial s_{21}} = -\frac{\frac{\partial \phi}{\partial v_2}\frac{\partial v_2}{\partial s_{21}}}{\partial \phi / \partial \kappa} = -\frac{\omega(\kappa^*)^{\alpha+1}}{(1-\omega)\mu_2 \partial \phi / \partial \kappa} < 0, \tag{21}$$

[11] Whilst calculating the signs of the derivatives (18–24) one should remember that in the range $(\overline{\kappa}, +\infty)$, so (particularly) also in κ^*, the derivative $\partial\phi/\partial\kappa$ has positive values.

$$\frac{\partial\kappa^*}{\partial\mu_1} = -\frac{\frac{\partial\phi}{\partial u_1}\frac{\partial u_1}{\partial\mu_1} + \frac{\partial\phi}{\partial v_1}\frac{\partial v_1}{\partial\mu_1}}{\partial\phi/\partial\kappa} = -\frac{(\kappa^*)^{\alpha}s_{11} + (\kappa^*)^{\beta}\frac{s_{12}(1-\omega)}{\omega}}{\mu_1^2\partial\phi/\partial\kappa} < 0, \quad (22)$$

$$\frac{\partial\kappa^*}{\partial\mu_2} = -\frac{\frac{\partial\phi}{\partial u_2}\frac{\partial u_2}{\partial\mu_2} + \frac{\partial\phi}{\partial v_2}\frac{\partial v_2}{\partial\mu_2}}{\partial\phi/\partial\kappa} = \frac{(\kappa^*)^{\beta+1}s_{22} + (\kappa^*)^{\alpha+1}\frac{s_{21}\omega}{1-\omega}}{\mu_2^2\partial\phi/\partial\kappa} > 0 \quad (23)$$

and:

$$\frac{\partial\kappa^*}{\partial\omega} = -\frac{\frac{\partial\phi}{\partial v_1}\frac{\partial v_1}{\partial\omega} + \frac{\partial\phi}{\partial v_2}\frac{\partial v_2}{\partial\omega}}{\partial\phi/\partial\kappa} = -\frac{(\kappa^*)^{\beta}\frac{s_{12}}{\mu_1\omega^2} + (\kappa^*)^{\alpha+1}\frac{s_{21}}{\mu_2(1-\omega)^2}}{\partial\phi/\partial\kappa} < 0. \quad (24)$$

The relationships (18–24) are economically interpreted that in the state of a long-run equilibrium of the analysed model of economic growth of capital-labour quotients k_1^*/k_2^*, that is κ^*, are the higher:

- the higher are the investment rates that are realised in economy 1 (i.e. s_{11} and s_{12}),
- the lower are the investment rates that are realised in economy 2 (i.e. s_{22} and s_{21}),
- the lower is the ratio of reduction in capital per worker in economy 1 (μ_1),
- the higher is the ratio of reduction in capital per worker in economy 2 (μ_2),
- the lower is the percentage of the workers in the economy of type 1(i.e. ω).

Moreover from equations (1) and from the considerations previously conducted it can be concluded that $k_i(t)\xrightarrow[t\to+\infty]{}k_i^*$ (for i = 1, 2), thus for any i, j = 1, 2 (when $j \neq i$) $y_i(t)\xrightarrow[t\to+\infty]{}y_i^* = \left(k_j^*\right)^{\beta}\left(k_i^*\right)^{\alpha}$. According to equation (2) it means that:

$$\forall\kappa^* > 0 \quad y^* = (\kappa^*)^{\alpha-\beta}, \quad (25)$$

where
$y^* = y_1^*/y_2^*$, so y^* denotes the relationship of labour productivity in economies of type 1 and 2 in the conditions of long-run equilibrium of the analysed model of economic growth. From relationship (25) it ensues also that:

$$\frac{dy^*}{d\kappa^*} = (\alpha-\beta)(\kappa^*)^{\alpha-\beta-1} > 0,$$

which means that:

$$\forall i, j = 1, 2 \ \operatorname{sgn}\frac{\partial y^\star}{\partial s_{ij}} = \operatorname{sgn}\frac{\partial \kappa^\star}{\partial s_{ij}}, \quad \operatorname{sgn}\frac{\partial y^\star}{\partial \mu_i} = \operatorname{sgn}\frac{\partial \kappa^\star}{\partial \mu_i}, \ \operatorname{sgn}\frac{\partial y^\star}{\partial \omega} = \operatorname{sgn}\frac{\partial \kappa^\star}{\partial \omega}. \quad (26)$$

From equations (26) it can be concluded that the monotonity of $y^\star$ in relation to the investment rates discussed, the ratios of reduction in capital per worker and the percentage of workers ω is the same as monotonity $\kappa^\star$ in relation to these variables.

Moreover it can be ensued from equations (12) and (25) that full convergence of capital-labour ratio and labour productivity between the types of economies analysed (which will occur when $\kappa^\star = y^\star = 1$)[12] will occur only if:

$$u_1 + v_1 = u_2 + v_2,$$

which equals the following relationship:

$$\frac{s_{11}\omega + s_{12}(1-\omega)}{\mu_1\omega} = \frac{s_{22}(1-\omega) + s_{21}\omega}{\mu_2(1-\omega)}. \quad (27)$$

It ensues from equation (27) that capital-labour level and labour productivity level in two types of economies will be equal when the sum of the investment rates realised in economy 1, weighted with weights $\frac{1}{\mu_1}$ and $\frac{1-\omega}{\mu_1\omega}$, equals the sum of the investment rates realised in economy 2, weighted with weights $\frac{1}{\mu_2}$ and $\frac{\omega}{\mu_2(1-\omega)}$.

[12] In the following considerations full convergence denotes a process in which the relationship between the same macroeconomic variable x in two types of economies will converge (in the long term) to 1. However when this variable converges to a value bigger than 1, then we speak about limited convergence. Thus, in the conditions of the full convergence of a variable, the following equality has to occur:

$\lim_{t\to+\infty}\frac{x_2(t)}{x_1(t)} = 1$, and by limited convergence: $\lim_{t\to+\infty}\frac{x_2(t)}{x_1(t)} = a > 1$, or: $\lim_{t\to+\infty}\frac{x_1(t)}{x_2(t)} = a > 1$.

Certainly in the state of equilibrium of the bipolar growth model analysed there has to occur a full or a limited convergence in the case of labour productivity as well as the capital-labour ratio, as $k_1(t)/k_2(t) \underset{t\to+\infty}{\to} \kappa^\star > 0$.

5. Parameter calibration and the selected states of long-run equilibrium

With the calibration of the values of the model of economic growth discussed the authors started with the attempt to determine the values of parameters α and β. To achieve this they began their considerations with the so-called Solow decomposition [1957] and gravity model of economic growth from the papers by Mroczek, Tokarski, and Trojak [2014] and Mroczek and Tokarski [2013]. Whereas it ensues from Solow decomposition and the theory of division by Clark that $\alpha \approx 1/3$, the calibration of the parameters of the gravity model of economic growth gives value β slightly smaller than $\alpha/3$. If it was assumed that $\alpha = 1/3$ and $\beta = 1/9$, it results from equation (2) that:

$$\frac{y_1(t)}{y_2(t)} = \left(\frac{k_1(t)}{k_2(t)}\right)^{2/9},$$

which by $k_1/k_2 = 5$ gives $y_1/y_2 \approx 1.430$. The relationship $y_1/y_2 \approx 1.430$ appears to be a greatly underestimated value. Moreover if it is assumed that k_j does not influence y_i (for $i \neq j$), that is $\beta = 0$, equation (2), where $\alpha = 1/3$, constitutes the following relationship:

$$\frac{y_1(t)}{y_2(t)} = \sqrt[3]{\frac{k_1(t)}{k_2(t)}}$$

then for $k_1/k_2 = 5$: $y_1/y_2 \approx 1.710$. This value also appears underestimated.[13]

Therefore for the calibration of the parameters of the growth model presented the authors decided to establish the value of flexibility α (with an additional assumption that $\beta = 0.3\alpha$) at the level that when $k_1/k_2 = 5$, the quotient of labour productivity y_1 and y_2 equals 3. Then, according to equation (2), the relationship: $\alpha = \frac{\ln 3}{0{,}7\ln 5} \approx 0.9752$,which means that $\beta \approx 0.2925$ thus $\alpha + \beta > 1$. Due to the fact that the sum of flexibilities α and β (according to the assump-

[13] Obviously assuming that in each of the analysed economies the function of labour productivity is given by the formula:
$y_i = A_i k_i^{\alpha}$, where $A_i > 0$ denotes total factor productivity in i-economy (for $i = 1, 2$), there is obtained:

$$\frac{y_1}{y_2} = \frac{A_1}{A_2}\left(\frac{k_1}{k_2}\right)^{\alpha}.$$

And then (even if $\alpha = 1/3$) total factor productivity A_i can be selected in a way that when $k_1/k_2 = 5$, the relationship y_1/y_2 equals 3 or 4. However in this case it has to be assumed that either the relationship A_1/A_2 (for some unknown reasons) is given for good, or – which is the case *implicit* in the models Mankiw-Romer-Weil [1992], Nonneman-Vanhoudt [1996] and in the growth model presented in this paper – the total factor productivity in each economy can be endogenised.

tions made in point 3) should be smaller than 1, therefore the authors decided to calibrate parameters α and β analogically, when $k_1/k_2 = 5$ and $y_1/y_2 = 2$, then, there was:[14]

$$\alpha = \frac{\ln 2}{0,7\ln 5} \approx 0.6153 \text{ and } \beta = 0.3\alpha \approx 0.1846$$

was achieved.

Having the values $\alpha \approx 0.6153$ and $\beta \approx 0.1846$ calibrated the authors assumed that the ratios of reduction in capital per worker in each of the types of economies discussed equals 6% (thus $\mu_1 = \mu_2 = 0.06$).

In the numeric simulations conducted it was analysed how the growth model presented in points 3–5 changes when the percentage of workers in rich economies (i.e. ω) equals changes of 10 percentage points, from 20% to 80% (variants 1ABC-7ABC in table).

Moreover it was assumed that the investments financed in economy i are realised in this economy at 90%, and in 10% in economy j (for $i \neq j$). Hence if s_i denotes the total percentage of investments in economy i in this production, $s_{ii} = 0.9s_i$ and $s_{ji} = 0.1s_i$ (for $j \neq i$). Furthermore the next ω values, three were analysed in the following variants.

Variant A, in which each of the economies is characterised by the investment rate equal to 22.5%, variant B – in which economy 1 has an investment rate equal to 25% and economy 2–20% and variant C – where $s_1 = 20\%$ and $s_2 = 25\%$.

It was also assumed that the starting values of the capital-labour ratio (i.e. The values of this variable in year $t = 1$) amount to $k_1 = 100$ and $k_2 = 20$. With such a configured combination of capital per worker, labour productivity in year $t = 1$ equals 29.56 in economy 1 and 14.78 in economy 2, whereas the capital output ratio (k_i/y_i for $i = 1, 2$) equals 3.383 and 1.353 respectively (long-run values of the capital output ratio in the variants of numeric simulations presented below are depicted in table).

The adoption of the above assumptions allowed the numeric simulations to be performed in two stages. In the former there was estimated κ^*, which solves equation (12), which also enabled – according to equation (25) – the estimation of γ^*. In the latter using the system of discrete differential equations:

$$\begin{cases} \Delta k_{1t} = s_{11}k_{1t-1}^{0.6153}k_{2t-1}^{0.1846} + s_{12}\dfrac{1-\omega}{\omega}k_{1t-1}^{0.1846}k_{2t-1}^{0.6153} - 0.06k_{1t-1}, \\ \Delta k_{2t} = s_{22}k_{2t-1}^{0.6153}k_{1t-1}^{0.1846} + s_{21}\dfrac{\omega}{1-\omega}k_{2t-1}^{0.1846}k_{1t-1}^{0.6153} - 0.06k_{2t-1}, \end{cases}$$

[14] The calibrated flexibility $\alpha \approx 0.6153$ is close to the estimated flexibilities of the labour productivity function in Polish provinces as presented in the paper by Mroczek and Tokarski [2013].

which is a counterpart of the system of differential equations (9), with the previously calibrated parameters and from the labour productivity function:

$$y_{it} = k_{it}^{0.6153}\, k_{jt}^{0.1846}$$

the authors determined the growth paths of the capital-labour ratio k_{it} and labour productivity y_{it} (where for i = 1, 2) in the following years t=1, 2, …, 200.

To calculate equation (12), i.e. κ^*, in the following simulation variants the authors used the following, simple numeric procedure. Firstly, the range (0.0005; 10) was divided into nearly 10 thousands ranges (κ_i, κ_{i+1}), where κ_1 = 0.0005, and for the next i $\kappa_{i+1} - \kappa_i$ = 0.0001. Secondly, there the values of function $\phi(\kappa)$ and $\left(\phi(\kappa_i)\right)^2$ in points $\kappa_{i\ were\ calculated}$. Thirdly, searching for the minimum of the value $\left(\phi(\kappa_i)\right)^2$ there was estimated κ^* (this calculation will be represented below as $\hat{\kappa}^*$). Subsequently from the estimation $\hat{\kappa}^*$ there was calculated $\hat{y}^* = (\hat{\kappa}^*)^{0.4307}$.

The estimations $\hat{\kappa}^*$, $\hat{y}^*$ as well as the results of the selected numeric simulations in variants 1ABC, 2ABC, …, 7ABC are presented in table. Moreover in the Figures 2–3 the growth paths of the selected macroeconomic variables in marginal variants, i.e., where: ω = 20% and $s_1 = s_2$ = 22.5% (Figure 2) and ω = 80% and $s_1 = s_2$ = 22.5% (Figure 3) were depicted.

From the selected results of the numeric simulations presented in table 1 the following conclusions can be drawn:

- In variants 1ABC when in the rich economies (of type 1) 20% of the workers will work and in poor economies (of type 2)[15] – 80%, thus poor economies will never catch up with the rich economies. The quotient of labour productivity will decrease from 2 to 1.124–1.362 and hence there will occur a limited convergence of this macroeconomic variable. In the case that both of types of economies are characterised by total investment rates equal to 22.5%, after 25 years the quotient of labour productivity will decrease to 1.352, after 50 years to 1.257, after 100 years to 1.233 (cf. also the Figures 2–3). In variant 1B (when richer economies have a total investment rates of 5 percentage points higher than poor economies) the relations of labour productivity will equal 1.476 after 25 years, 1.389 after 50 years and 1.364 after 100 years. In variant 1C (when poor economies are characterised by higher total investment rates) the discussed quotients will be as follows: 1.245 (after 25 years), 1.147 (after 50 years) and 1.125 (after 100 years).
- Also when the percentage of the workers in rich countries amounts to 30% (variants 2ABC) poor economies will never catch up with rich economies. In these cases long-term relationships between product per worker will be between 1.013 (variant 2C) and 1.258 (variant 2B), which denotes that also

[15] In the further part of point 6, economies of type 1 are called relatively rich economies (or initially rich), and the ones of type 2 – relatively poor (or initially poor).

then there will occur a limited convergence of labour productivity. In variant 2A, after 25 years, labour productivity in rich economies will be 27.1% higher than in poor economies, after 50 years – 15.9% higher and after 100 years – 12.7% higher. In variant 2B production per worker in rich economies will be 39.4% higher than in poor economies (after 25 years), 29.1% (after 50 years) and 26.1% (after 100 years). In variant 2C in rich economies production per worker will be higher than in poor economies at 16.3% (after 25 years), 4.6% (after 50 years) and 1.5% (after 100 years).

- Likewise in the variants described above, as also in variants 3AB full convergence of capital-labour ratios and of labour productivity will never occur. In variant 3A, long-term labour productivity in rich economies should be 5.5% higher than in poor economies, and in variant 3B – 18.6% higher. After 25, 50 and 100 years the quotients of labour productivity will equal respectively 1.215; 1.094 and 1.059 (where the total investments rates are equal and amount to 22.5%) or 1.331; 1.222 and 1.189 (where rich economies invest 25% of their product and the poor – 20%).

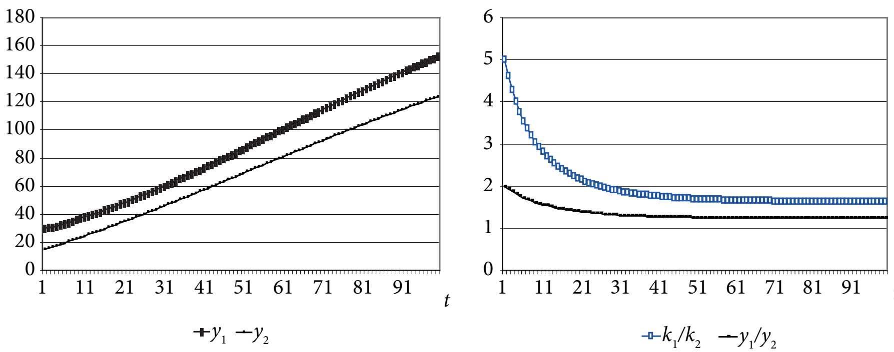

Figure 2. The simulation of the paths of labour productivity growth y_1 and y_2 and the relationship k_1/k_2 and y_1/y_2 where $\omega = 0.2$ and $s_1 = s_2 = 0.225$

- If the percentage of the workers in rich economies amounts to 40% these economies invest 20% of the product and the poor – 25%, long-run labour productivity in the initially rich economies will amount to 94.3% of labour productivity in the initially poor economies. Initially poor economies will catch up with rich economies after 42 years. After 25 years in this variant the relationship y_1/y_2 will amount to 1.110, after 50 years – 0.982, and after 100 years – 0.946.
- In variant 4A (i.e. when both types of economies absorb 50% of the total number of workers and when they are characterised by 22.5% investment rates) poor economies will permanently endeavour to catch up with rich economies as regards the capital-labour ratio and also labour productivity

to catch up with them in the long run. After 25 years, labour productivity in the rich economies will be 16.3% higher than in poor economies, after 50 years 4.0% higher and after 100 years only 0.3% higher.

- If the number of workers in rich economies equals the number of workers in poor economies the total investment rate in rich economies amounts to 25% and in the poor 20%, the long-run relationship of labour productivity y_1/y_2 will amount to 1.124. After 25 years labour productivity in rich economies will be 27.1% higher than in poor economies, after 50 years 16.0% higher and after 100 years 12.7% higher.
- If the investment rate (when ω = 50%) in economies of type 1 amounts to 20%, in economies of type 2–25%, after 32 years the initially poor economies will surpass the initially rich economies and in the long run the relationship y_1/y_2 will tend to 0.890. In this case after 25 years the quotient y_1/y_2 will equal 1.064, after 25 years 0.932, and after 100 years 0.894.
- In variant 5B poor economies will only endeavour to catch up with rich economies so that in long run labour productivity in rich economies is higher than in poor economies at only 6.1%. After 25 years the production per worker in rich economies will be higher than in poor economies by 20.4%, after 50 years by 9.4% and after 100 years by 6.3%.

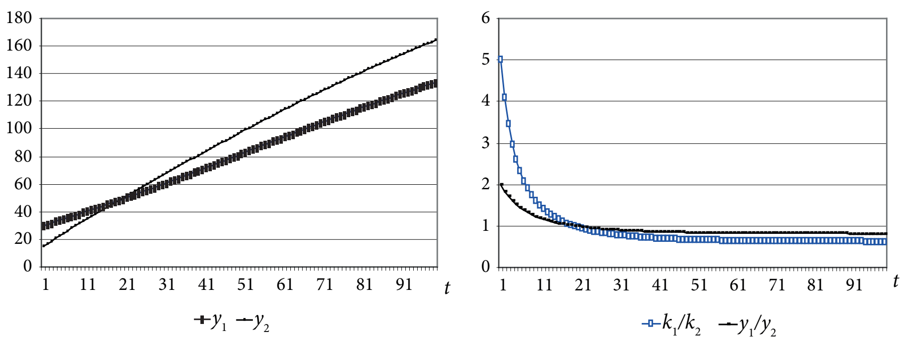

Figure 3. The simulation of the paths of labour productivity growth y_1 and y_2 and the relationship k_1/k_2 and y_1/y_2 where ω = 0.8 and $s_1 = s_2$ = 0.225

- When the percentage of the workers in the initially rich economies amounts to 70%, the initially poor economies will surpass the initially rich economies after 29 years (variant 6A), 65 years (6B) or 21 years (6C). Long-run relationships of labour productivity will be then equal to 0.889; 0.987 or 0.795. After 25, 50 and 100 years the quotients y_1/y_2 will equal: 1.039; 0.924 and 0.892 (in variant 6A) or 1.119; 1.017 and 0.990 (variant 6B) or 0.962; 0.835 and 0.798 (6C).
- When ω = 80%, the initially poor economies will surpass the initially rich economies after 19 years (variant 7A, illustrated in the Figure 3), 25 years

The results of the numeric simulations

Variant	ω	s_1	s_{11}	s_{12}	s_2	s_{22}	s_{21}	The estimated long-run relations		Long-run capital-output ratio in economy[a]		After how many years economy 2 will catch up with economy 1?	Estimation error κ^* (measured $\phi(\hat{\kappa}^*)$)
	in %							Capital-labour ratio ($\hat{\kappa}^*$)	Labour productivity ($\hat{y}^*$)	1	2		
1A		22.50	20.25	2.25	22.50	20.25	2.25	1.620	1.231	4.396	3.340	never	$6.57*10^{-5}$
1B	20	25.00	22.50	2.00	20.00	18.00	2.50	2.050	1.362	4.535	3.013	never	$1.64*10^{-5}$
1C		20.00	18.00	2.50	25.00	22.50	2.00	1.310	1.124	4.281	3.670	never	$-1.35*10^{-4}$
2A		22.50	20.25	2.25	22.50	20.25	2.25	1.313	1.125	3.985	3.412	never	$-1.25*10^{-4}$
2B	30	25.00	22.50	2.00	20.00	18.00	2.50	1.704	1.258	4.198	3.099	never	$8.13*10^{-5}$
2C		20.00	18.00	2.50	25.00	22.50	2.00	1.030	1.013	3.795	3.732	never	$6.52*10^{-5}$
3A		22.50	20.25	2.25	22.50	20.25	2.25	1.133	1.055	3.756	3.498	never	$3.90*10^{-6}$
3B	40	25.00	22.50	2.00	20.00	18.00	2.50	1.486	1.186	4.013	3.202	never	$6.07*10^{-5}$
3C		20.00	18.00	2.50	25.00	22.50	2.00	0.872	0.943	3.519	3.803	42	$5.28*10^{-5}$
4A		22.50	20.25	2.25	22.50	20.25	2.25	1	1	3.608	3.608	never[b]	0
4B	50	25.00	22.50	2.00	20.00	18.00	2.50	1.311	1.124	3.894	3.337	never	$8.80*10^{-5}$
4C		20.00	18.00	2.50	25.00	22.50	2.00	0.763	0.890	3.337	3.894	32	$4.71*10^{-5}$

5A		22.50	20.25	2.25	22.50	20.25	2.25	0.882	0.947	3.503	3.762	44	$-6.15*10^{-5}$
5B	60	25.00	22.50	2.00	20.00	18.00	2.50	1.147	1.061	3.809	3.524	never	$1.19*10^{-4}$
5C		20.00	18.00	2.50	25.00	22.50	2.00	0.673	0.843	3.208	4.019	26	$1.04*10^{-4}$
6A		22.50	20.25	2.25	22.50	20.25	2.25	0.761	0.889	3.423	3.998	29	$1.08*10^{-5}$
6B	70	25.00	22.50	2.00	20.00	18.00	2.50	0.971	0.987	3.744	3.807	65	$5.59*10^{-5}$
6C		20.00	18.00	2.50	25.00	22.50	2.00	0.587	0.795	3.109	4.211	21	$9.65*10^{-5}$
7A		22.50	20.25	2.25	22.50	20.25	2.25	0.617	0.812	3.357	4.418	19	$1.95*10^{-5}$
7B	80	25.00	22.50	2.00	20.00	18.00	2.50	0.763	0.890	3.689	4.303	25	$-1.25*10^{-4}$
7C		20.00	18.00	2.50	25.00	22.50	2.00	0.488	0.734	3.029	4.559	16	$8.28*10^{-6}$

[a] Long-run capital output ratio in economy 1 or 2 is understood as capital output ratio in the horizon of 200 years of numeric simulations.
[b] But (where $t \to +\infty$) labour productivity in both economies will be convergent.

(7B) or 16 years (7C). In the long run labour productivity in initially rich economies will equal 81.2%, 89.0% or 73.4% of labour productivity in initially poor economies. After 5, 50 and 100 years the quotients y_1/y_2 will equal: 0.942; 0.840 and 0.814 (variant 7A) or 1.001; 0.912 and 0.892 (7B) or 0.882; 0.768 and 0.737 (7C).

Conclusions

The following conclusions can be drawn from the above considerations:

1. In this model it is assumed that between two types of economies (relative rich and relative poor economies) investment flows occur. Thus the process of physical capital accumulation in each of the economies results from domestic as well as foreign investment. Moreover it is also assumed that the level of labour productivity in a particular economy depends not only on the level of its capital-labour ratio but also the level of the capital-labour ratio of the economy with which it is connected by certain economic dependencies.
2. The system of differential equations as discussed in this model has exactly one non-trivial steady states. By means of the Grobman-Hartman theorem it was depicted that this point is an asymptotically stable point. Therefore it determines the state of the long-run equilibrium of the economic growth model discussed.
3. The capital-labour ratio (k_1^*/k_2^*) and the labour productivity ratio (y_1^*/y_2^*) in the state of long-run equilibrium of a bipolar model of economic growth depend on investment rates, the ratios of reduction in capital per worker and the percentage of workers in each economy. The higher the investment rates realised in a rich economy and the ratio of reduction in capital per worker in a poor economy and the lower the investment rates realised in a poor economy, the ratio of reduction in capital per worker in a poor economy and the percentage of workers in a rich economy, the higher are the quotients of capital-labour ratios (k_1^*/k_2^*) and labour productivity ratios (y_1^*/y_2^*) in the state of the long-term equilibrium of the model.
4. In order to conduct numeric simulations of the state of the long-run equilibrium, the parameters of the model were calibrated. Parameter α, i.e., the flexibility of labour productivity in the first (in the second) economy in relation to the capital-labour ratio in the first (the second) economy was determined at the level equal to approximately 0.6153, whereas the parameter β, i.e., the flexibility of the total factor productivity in the first (the second) economy in relation to the capital-labour ratio in the second (the first) economy was calibrated at the level of 0.1846. It was also assumed that the ratios of reduction in capital per worker in each of the types of economies analysed equal 6%.

5. In the numeric analysis conducted the following assumption was made: the starting level of capital-labour ratio in the rich economy is five times higher than in the poor economy, moreover in each of the economies 90% of the self-financed investments are realised inside the country and only 10% abroad. Numeric simulations were conducted in 7 variants of the percentage of workers in rich economies (from 20% to 80% every 10 percentage points). Additionally for each of the above variants there three of the following combinations of investment rates were considered: the first – investment rates equal 22.5% in each of the economies; the second – investment rate in rich economies amounts to 25% and in poor economies – 20%, and the third – the investment rate in poor economies 25% and in the rich – 20%.
6. In the first two variants (i.e., when in rich economies 20% and 30% of the workers actually work), irrespective of the accepted combination of investment rates, poor economies will never catch up with rich economies. When the percentage of workers in rich countries amounts to 40% and 50%, poor countries can catch up with rich economies only in the third set of investment rates (i.e., investing 5 percentage points more than rich countries). When 60% of the workers work in rich countries, poor economies can catch up with rich economies only when their investment rates are equal or higher than these rates in rich countries.

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