Economics and Business Review, Vol. 6 (20), No. 2, 2020: 66-80
DOI: 10.18559/ebr.2020.2.5

Almost "very strong" multilane turnpike effect in a non-stationary Gale economy with a temporary von Neumann equilibrium and price constraints[1]

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Abstract: Mathematical models of economic dynamics and growth are usually expressed in terms of differential equations/inclusions (in the case of continuous time) or difference equations/inclusions (if discrete time is assumed).[3] This class of models includes von Neumann-Leontief-Gale type dynamic input-output models to which the paper refers. The paper focuses on the turnpike stability of optimal growth processes in a Gale non-stationary economy with discrete time in the neighbourhood of von Neumann dynamic equilibrium states (so-called growth equilibrium). The paper refers to Panek (2019, 2020) and shows an intermediate result between the *strong* and *very strong* turnpike theorem in the non-stationary Gale economy with changing technology assuming that the prices of temporary equilibrium in such an economy (so-called von Neumann prices) do not change rapidly. The aim of the paper is to bring mathematical proof that the introduction of these assumptions making the model more realistic does not change its asymptotic (*turnpike-like*) properties.

Keywords: non-stationary Gale economy, temporary von Neumann equilibrium, multilane production turnpike.

JEL codes: C62, C67, O41, O49.

Introduction

In the paper Panek (2018) three theorems about turnpikes[4] in the non-stationary Gale economy with a technological limit are presented, where a single pro-

[1] Article received 15 January 2020, accepted 8 June 2020.

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[3] See e.g. Acemoglu (2009), Deng, Fujio, and Khan (2019), Galor (2010), Guzowska (2018), Makarov, Rubinov (1977), McKenzie (2005), Mitra and Nishimura (2009), Panek (2003), Zalai (2004).

[4] „Weak", „strong" and „very strong". The basic of turnpike theory and its main theorems can be found in Makarov and Rubinov (1977), McKenzie (2005), Nikaido (1968), Panek (2003).

duction turnpike was replaced with a bundle of turnpikes (so-called multilane turnpike). In the proofs the assumption that a production technology available in the economy at period will be available also in the next period was used. This assumption may be justified in short time intervals but it is difficult to explain in longer term. Such a condition does not have to be met in the growth model presented in Panek (2019, 2020). In the first article a non-stationary economy with changing technology and increasing production efficiency over time is dealt with and in the second the proof of a "weak" multiline turnpike theorem in the Gale economy with changing technology and monotonically changing prices is presented.

The main result of this work is a quasi "very strong" turnpike theorem (Th. 3) in which, based on the idea contained in Panek (2014), it is assumed that among assumptions analogous to Panek (2020) an extra (rather weak) condition is satisfied: in long periods (horizons of the economy) temporary von Neumann equilibrium prices have restricted dynamics (they do not change rapidly) in the final phase of growth.

The rest of the paper is as follows. In Section 1 a non-stationary Gale-type model of the economy, multilane production turnpike and a temporary von Neumann equilibrium is presented. In Section 2 feasible and stationary growth processes are defined. The main result of Section 3 is a proof of a quasi "very strong" turnpike theorem (Th. 3). The paper closes with conclusions.

1. The model. multilane turnpike and temporary von Neumann equilibrium[5]

Let n denote a number of goods in the economy. Time t is discrete, $t = 0, 1, \ldots$. Let $x(t) = (x_1(t), \ldots, x_n(t))$ denote a non-negative vector of goods consumed during period t (vector of consumption or inputs). Let $y(t) = (y_1(t), \ldots, y_n(t))$ denote a non-negative vector of goods produced in this period from the inputs $x(t)$ (outputs or production vector). If from the inputs $x(t)$ one can produce $y(t)$, then an ordered pair $(x(t), y(t))$ determines a technologically admissible production process in time t.[6] A non-empty set $Z(t) \subset R^{2n}_+$ of all technologically admissible production processes in period t is called a production space (technological set) in period t. Written as $(x, y) \in Z(t)$ (or equivalently $(x(t), y(t)) \in Z(t)$) if in the period t one can produce from the inputs $x = x(t)$ a production vector $y = y(t)$. Sets $Z(t)$, $t = 0, 1, \ldots$, are called Gale production spaces if they satisfy the following conditions:

[5] The model described below is related to Panek (2020).

[6] By R^{2n}_+ a non-negative orthant of an $2n$-dimensional vector space is denoted, $R^{2n}_+ = \{z \in R^{2n}_+ | z \geqq 0\}$. If $x, y \in R^n$, then $x \geqq y$ means that $\forall i(x_i \geq y_i)$. Notation $x \geq y$ is equivalent to $x \geqq y$ and $x \neq y$.

(G1) $\forall(x^1, y^1) \in Z(t)\ \forall(x^2, y^2) \in Z(t)\ \forall\ \lambda_1, \lambda_2 \geq 0(\lambda_1(x^1, y^1) + \lambda_2(x^2, y^2) \in Z(t))$

(proportionality of input/output and additivity of production).

(G2) $\forall(x, y) \in Z(t)\ (x = 0 \Rightarrow y = 0)$

("no cornucopia" condition).

(G3) $\forall(x, y) \in Z(t)\ \forall x' \geqq x\ \forall\ 0 \leqq y' \leqq y\ ((x', y') \in Z(t))$

(possibility of wasting inputs and / or outputs).

(G4) Sets $Z(t)$ are closed in R^{2n}_+.

Gale's production spaces are convex cones and are closed in R^{2n}_+ with vertices in the origin of the coordinate system. The cone can change in time in accordance with changes in production technology of the economy. All Gale's production spaces share the property such that if in certain period t there is $(x, y) \in Z(t)$ and $x = 0$, then $y = 0$. The rest of the article is concerned with in the non-zero production processes $(x, y) \in Z(t)\backslash\{0\}$, $t = 0, 1, \ldots$ A number

$$\alpha(x, y) = \max\{\alpha | \alpha x \leqq y\}$$

is called the technological efficiency rate of the process (x, y). Under conditions **(G1)-(G4)**, function $\alpha(\cdot)$ is positively homogenous of degree zero on a set $Z\backslash\{0\}$ and

$$\exists(\bar{x},\bar{y}) \in Z(t)\backslash\{0\}\left(\alpha(\bar{x},\bar{y}) = \max_{(x,y)\in Z(t)\backslash\{0\}} \alpha(x,y) = \alpha_{M,t} \geq 0\right).$$

Cf. Takayama (1985, Th. 6.A.1 – replace $Z(t)$ with T and $\alpha_{M,t}$ with $\hat{\alpha}$). A process $(\bar{x},\bar{y}) = (\bar{x}(t),\bar{y}(t))$ is called an optimal production process and $\alpha_{M,t}$ an optimal efficiency rate in the the non-stationary Gale economy in period t. In each moment of time the optimal production process is determined by a multiplication by a positive constant (up to the structure):

$$\forall\lambda > 0\left(\alpha(\bar{x}(t),\bar{y}(t)) = \alpha(\lambda\bar{x}(t),\lambda\bar{y}(t)) = \alpha_{M,t} \geq 0\right).$$

To exclude the uninteresting and unrealistic case $\alpha_{M,t} = 0$ it is assumed that

(G5) $$\forall t\, \forall i\, \exists(x^i, y^i) \in Z(t)\backslash\{0\}(y^i_i > 0)$$

(it is possible to produce any good in the economy at any moment of time).[7] Indeed, let us fix a moment t. Let $\left(x^i(t), y^i(t)\right) \in Z(t)\setminus\{0\}$ denote an admissible production process, such that $y_i^i(t) > 0$, $i = 1, \ldots, n$. Due to **(G1)** the process

$$\left(x(t), y(t)\right) = \sum_{i=1}^{n}\left(x^i(t), y^i(t)\right) \in Z(t)\setminus\{0\}.$$

is admissible and $y(t) > 0$, so $\alpha(x(t), y(t)) > 0$. Hence,

$$\alpha_{M,t} = \max_{(x,y)\in Z(t)\setminus\{0\}} \alpha(x, y) \geq \alpha\left(x(t), y(t)\right) > 0.$$

The set

$$Z_{opt}(t) = \left\{(\overline{x}, \overline{y}) \in Z(t)\setminus\{0\} \,|\, \alpha(\overline{x}, \overline{y}) = \alpha_{M,t} > 0\right\}$$

is an collection of all optimal production processes in the non-stationary Gale economy in the period t. Sets $Z_{opt}(t) \subset Z(t)\setminus\{0\}$, $t = 0, 1, \ldots$, are cones contained in $R_+^{2n}\setminus\{0\}$, with no element at 0. If $\left(\overline{x}(t), \overline{y}(t)\right) \in Z_{opt}(t)$, then a vector $s(t) = \frac{\overline{y}(t)}{\left\|\overline{y}(t)\right\|}$ characterizes the structure of production in the optimal process $\left(\overline{x}(t), \overline{y}(t)\right)$.[8] Condition $\left(\overline{x}(t), \overline{y}(t)\right) \in Z_{opt}(t)$ combined with **(G3)** implies that:

$$\left(\overline{x}(t), \alpha_{M,t}\overline{x}(t)\right) \in Z_{opt}(t) \text{ and } \left(\overline{y}(t), \alpha_{M,t}\overline{y}(t)\right) \in Z_{opt}(t).$$

If the conditions **(G1)-(G5)** hold then sets

$$S(t) = \left\{ s(t) \,|\, \exists\left(\overline{x}(t), \overline{y}(t)\right) \in Z_{opt}(t)\left(s(t) = \frac{\overline{y}(t)}{\left\|\overline{y}(t)\right\|}\right)\right\}, t = 0, 1, \ldots,$$

consisting of production structure vectors in all optimal processes for each particular time period $t \geq 0$ are compact and convex; Panek (2016, Th. 2; under substitution: $S(t)$ in place of S). A ray

$$N_s^t = \{\lambda s | \lambda > 0\} \subset R_+^n,$$

[7] In the proof of a "quasi very strong" turnpike theorem (theorem 3) only a weaker assumption is necessary: every good can be produced from a particular period of time $\overline{t} \geq 0$.

[8] If $a \in R^n$, then $\|a\| = \sum_{i=1}^{n}|a_i|$, $\frac{a}{\|a\|} = \left(\frac{a_1}{\|a\|}, \frac{a_2}{\|a\|}, \ldots, \frac{a_n}{\|a\|}\right)$.

where $s = s(t) \in S(t)$, is called a single turnpike (von Neumann's ray) in the non-stationary Gale economy at period t. A set

$$\mathbb{N}^t = \bigcup_{s \in S(t)} N_s^t = \left\{ \lambda s \mid \lambda > 0, s \in S(t) \right\}$$

is called a multiline turnpike in the non-stationary Gale economy at period t. Multiline turnpikes $\mathbb{N}^t$ in all periods $t \geq 0$ are convex cones contained in R_+^n, without the zero vector. If in a certain process $(x, y) \in Z(t) \setminus \{0\}$ at period t he inputs $\frac{x}{\|x\|}$ or outputs structure $\frac{y}{\|y\|}$ deviate from the turnpike structure, then the technological efficiency of such a process is smaller than optimal:

$$\frac{x}{\|x\|} \notin S(t) \vee \frac{y}{\|y\|} \notin S(t) \Rightarrow \alpha(x, y) < \alpha_{M,t}.^{9} \tag{1}$$

Let $p(t) = \left(p_1(t), \ldots, p_n(t)\right) \geq 0$ denote the price vector of goods produced at period t. Formula $\left\langle p(t), x(t) \right\rangle = \sum_{i=1}^{n} p_i(t) x_i(t)$ denotes the amount of expenditures $\left\langle p(t), y(t) \right\rangle = \sum_{i=1}^{n} p_i(t) y_i(t)$ and denotes the amount of production in the process $\left(x(t), y(t)\right) \in Z(t) \setminus \{0\}$. A ratio

$$\beta\left(x(t), y(t), p(t)\right) = \frac{\left\langle p(t), y(t) \right\rangle}{\left\langle p(t), x(t) \right\rangle} \geq 0$$

($\left\langle p(t), x(t) \right\rangle \neq 0$) is called the economic efficiency rate of the process ($\left(x(t), y(t)\right)$ with prices $p(t)$).

☐ **Theorem 1.** Assume that conditions **(G1)-(G5)** are satisfied. Then for each t a price vector exists $\overline{p}(t) \geq 0$, such that

$$\forall (x, y) \in Z(t) \left(\left\langle \overline{p}(t), y \right\rangle \leq \alpha_{M,t} \left\langle \overline{p}(t), x \right\rangle \right).$$

Proof. Cf. Panek (2020, Th. 1). ■

It follows from the definition of the optimal production process that:

$$\forall \left(\overline{x}(t), \overline{y}(t)\right) \in Z_{opt}(t) \subset Z(t) \setminus \{0\} \left(\alpha\left(\overline{x}(t), \overline{y}(t)\right) = \max_{\substack{\alpha x \leqq y \\ (x,y) \in Z(t) \setminus \{0\}}} \alpha = \alpha_{M,t} > 0 \right),$$

[9] Panek (2018, Lemma 1).

i.e. $\forall\left(\overline{x}(t),\overline{y}(t)\right)\in Z_{opt}(t)\left(\alpha_{M,t}\overline{x}(t)\leqq\overline{y}(t)\right)$, thus:

$$\left\langle\overline{p}(t),\overline{y}(t)\right\rangle\geq\alpha_{M,t}\left\langle\overline{p}(t),\overline{x}(t)\right\rangle\geq 0.$$

Hence according to theorem 1 the result is:

$$\left\langle\overline{p}(t),\overline{y}(t)\right\rangle\leq\alpha_{M,t}\left\langle\overline{p}(t),\overline{x}(t)\right\rangle,$$

and finally:

$$\forall\left(\overline{x}(t),\overline{y}(t)\right)\in Z_{opt}(t)\left(\left\langle\overline{p}(t),\overline{y}(t)\right\rangle=\alpha_{M,t}\left\langle\overline{p}(t),\overline{x}(t)\right\rangle\geq 0\right).\tag{2}$$

It is assumed that the economy cannot reach a maximal economic efficiency if it does not have maximal technological efficiency:

(G6)

$$\forall t\geq 0\,\forall(x,y)\in Z(t)\setminus\{0\}\left(\alpha(x,y)<\alpha_{M,t}\Rightarrow 0\leq\beta\left(x,y,\overline{p}(t)\right)=\frac{\left\langle\overline{p}(t),y\right\rangle}{\left\langle\overline{p}(t),x\right\rangle}<\alpha_{M,t}\right).$$

This is a realistic assumption. The following theorem is a consequence of the discussion above.

☐ **Theorem 2.** If conditions **(G1)-(G6)** hold, then
$\forall t\,\forall(\overline{x},\overline{y})\in Z_{opt}(t)\subset Z(t)\setminus\{0\}$:

$$\alpha_{M,t}\overline{x}\leqq\overline{y},\tag{3}$$

$\forall t\,\exists\overline{p}(t)\geq 0$:

$$\forall(x,y)\in Z(t)\setminus\{0\}\left(\left\langle\overline{p}(t),y\right\rangle\leq\alpha_{M,t}\left\langle\overline{p}(t),x\right\rangle\right),\tag{4}$$

and $\forall t\,\forall(\overline{x},\overline{y})\in Z_{opt}(t)$:

$$\left\langle\overline{p}(t),\overline{y}\right\rangle>0.\tag{5}$$

Proof. Condition (3) follows directly from the definition of the optimal production $(\overline{x},\overline{y})\in Z_{opt}(t)\subset Z(t)\setminus\{0\}$. Condition (4) is a consequence of theorem 1. To prove condition (5) any optimal process are considered $(\overline{x},\overline{y})\in Z_{opt}(t)$. It follows from (2) that

$$\left\langle\overline{p},\overline{y}\right\rangle\leq\alpha_{M,t}\left\langle\overline{p},\overline{x}\right\rangle\geq 0,$$

and the definition of an optimal process implies:

$$\exists k \left(\alpha(\overline{x},\overline{y}) = \min_i \frac{\overline{y}_i}{\overline{x}_i} = \frac{\overline{y}_k}{\overline{x}_k} = \alpha_{M,t} > 0 \right),$$

hence $\overline{y}_k > 0$. Let $\tilde{x} = \overline{x} + e^k$, where $e^k = (0, \ldots, 1, \ldots, 0)$ is a n-dimensional vector with k-th coordinate equal to 1. Since $(\overline{x},\overline{y}) \in Z_{opt}(t) \subset Z(t) \setminus \{0\}$, the pair $(\tilde{x}, \overline{y}) \in Z(t)\setminus\{0\}$ is an admissible process at period t (which follows from (**3**)), but is not optimal because $\alpha(\tilde{x}, \overline{y}) < \alpha_{M,t}$. **(G6)** implies:

$$\beta\left(\tilde{x},\overline{y},\overline{p}(t)\right) < \alpha_{M,t}. \tag{6}$$

Suppose that $\left\langle \overline{p}(t),\overline{y} \right\rangle = 0$, then $\overline{p}_k(t) = 0$, hence $\left\langle \overline{p},\tilde{x} \right\rangle = \left\langle \overline{p},\overline{x} \right\rangle$. Moreover conditions (2) and (3) imply that $\left\langle \overline{p},\overline{x} \right\rangle = 0$ and:

$$\beta(\tilde{x},\overline{y},\overline{p}) = \frac{\left\langle \overline{p},\overline{y} \right\rangle}{\left\langle \overline{p},\tilde{x} \right\rangle} = \frac{\left\langle \overline{p},\overline{y} \right\rangle}{\left\langle \overline{p},\overline{x} \right\rangle} = \frac{0}{0},$$

which contradicts (6) and concludes the proof. ■

Theorem 1 implies that (assuming $\left\langle \overline{p}(t),x(t) \right\rangle \neq 0$):

$$\forall \left(x(t),y(t)\right) \in Z(t) \setminus \{0\} \left(\beta\left(x(t),y(t),\overline{p}(t)\right) = \frac{\left\langle \overline{p}(t),y(t) \right\rangle}{\left\langle \overline{p}(t),x(t) \right\rangle} \leq \alpha_{M,t} \right),$$

i.e. the economic efficiency of any production process at period never exceeds the optimal technological production efficiency which (at that period) the economy can reach. On the other hand theorem 2 implies

$$\begin{aligned} &\forall t \, \forall \left(\overline{x}(t),\overline{y}(t)\right) \in Z_{opt}(t): \\ &\beta\left(\overline{x}(t),\overline{y}(t),\overline{p}(t)\right) = \frac{\left\langle \overline{p}(t),\overline{y}(t) \right\rangle}{\left\langle \overline{p}(t),\overline{x}(t) \right\rangle} = \max_{(x,y)\in Z(t)\setminus\{0\}} \beta\left(x,y,\overline{p}(t)\right) = \\ &= \alpha\left(\overline{x}(t),\overline{y}(t)\right) = \alpha_{M,t} > 0. \end{aligned} \tag{7}$$

It can be stated a triple $\left\{\alpha_{M,t}, \left(\overline{x}(t),\overline{y}(t)\right), \overline{p}(t)\right\}$ which satisfies (3)-(5) defines a temporary von Neumann equilibrium at period t. Prices $\overline{p}(t)$ are called (temporary) equilibrium prices. In a Gale economy which satisfies conditions **(G1)**-**(G6)** the temporary equilibrium always exists at each period t. It consists of a triple $\left\{\alpha_{M,t},\left(\overline{x}(t),\overline{y}(t)\right),\ \overline{p}(t)\right\}$ with the optimal technological efficiency rate $\alpha_{M,t}$, equilibrium prices $\overline{p}(t)$ and any optimal production process $\left(\overline{x}(t),\overline{y}(t)\right)\in Z_{opt}(t)$.[10]

Production processes $(x, y) \in Z(t)\backslash\{0\}$ as well as inputs x and/or outputs y, on a multiline turnpike $\mathbb{N}^t$ in the Gale economy are well-defined up to structure (multiplication by a positive constant). That provides a way to define an (angular) measure of distance of the input vector x from the turnpike $\mathbb{N}^t$:

$$d(x,\mathbb{N}^t)=\inf_{x'\in\mathbb{N}^t}\left\|\frac{x}{\|x\|}-\frac{x'}{\|x'\|}\right\|.^{11} \tag{8}$$

□ **Lemma 1.** Assume that conditions **(G1)-(G6)** hold, then

$$\forall\varepsilon>0\ \forall t\geq 0\ \exists\ \delta_{\varepsilon,t}\in(0,\alpha_{M,t})\ \forall(x,y)Z(t)\backslash\{0\}:$$
$$d(x,\mathbb{N}^t)\geq\varepsilon\Rightarrow\left\langle\overline{p}(t),y\right\rangle-(\alpha_{M,t}-\delta_{\varepsilon,t})\left\langle\overline{p}(t),x\right\rangle\leq 0.$$

Proof is similar to the proof of lemma 2 in Panek (2019; under the substitution of $\mathbb{N}^t$, $\delta_{\varepsilon,t}$, $\overline{p}(t)$ in place of $\mathbb{N},\delta_{\varepsilon,t_1},\overline{p}(t_1)$.[12] ■

The lemma implies for any period t that if the inputs structure in an admissible process $(x, y) = (x(t), y(t)) \in Z(t)\backslash\{0\}$ differs by $\varepsilon > 0$ (in the sense of metric (8)) from the turnpike structure, then the economic efficiency of such a process is lower than maximum (which equals $\alpha_{M,t}$; cf. (7)) by at least $\delta_{\varepsilon,t}\in(0,\alpha_{M,t})$:

$$\beta\left(x,y,\overline{p}(t)\right)=\frac{\left\langle\overline{p}(t),y\right\rangle}{\left\langle\overline{p}(t),x\right\rangle}\leq\alpha_{M,t}-\delta_{\varepsilon,t}.$$

Conditions (1) and **(G6)** imply that at any period the following implication holds:

$$x(t)\notin\mathbb{N}^t\vee y(t)\notin\mathbb{N}^t\Rightarrow\alpha\left(x(t),y(t)\right)<\alpha_{M,t}\Rightarrow\beta\left(x(t),y(t),\overline{p}(t)\right)=$$
$$=\frac{\left\langle\overline{p}(t),y(t)\right\rangle}{\left\langle\overline{p}(t),x(t)\right\rangle}<\alpha_{M,t},$$

[10] In the temporary von Neumann equilibrium at period t. the economic efficiency matches the technological efficiency at the maximal level which is possible to reach by the economy at that period. Prices and optimal production processes in the von Neumann equilibrium are well-defined up to structure.

[11] A distance of the production vector y from the multiline turnpike is analogously defined.

[12] This is a variant of the lemma of Radner (1961) adjusted to the specifics of a non-stationary Gale economy.

equivalently:

$$\beta\left(\overline{x}(t),\overline{y}(t),\overline{p}(t)\right)=\alpha_{M,t}\Rightarrow\alpha\left(\overline{x}(t),\overline{y}(t)\right)=\alpha_{M,t}\Rightarrow\ \overline{x}(t)\in\mathbb{N}^t\wedge\ \overline{y}(t)\in\mathbb{N}^t.$$

The conclusion can be drawn that at each period the non-stationary Gale economy satisfying conditions **(G1)-(G6)** reaches its maximal (economic, technological) efficiency only on the turnpike $\mathbb{N}^t$.

Lemma 1 plays a key role in the proof of the almost "very strong" turnpike theorem (theorem 3). However the condition **(G6)** does not exclude the following implausible scenario: economic efficiency of a certain process $(x(t), y(t))$ grows with time $t \to +\infty$, reaching the maximum in the limit despite the fact that the inputs structure deviates from the turnpike structure by a positive constant $\varepsilon > 0$ which is independent of time. The following condition will exclude such a situation:[13]

(**G7**) $$\forall\varepsilon>0\,\exists\nu_\varepsilon>0\,\forall t\left(\frac{\delta_{\varepsilon,t}}{\alpha_{M,t}}\geq\nu_\varepsilon\right).$$

2. Dynamics. Feasible and stationary growth processes

$T = \{0, 1, \ldots, t_1\}$ denotes an interesting fixed horizon of an economy $t_1 < +\infty$. It is assumed that the economy is closed, i.e. the expenditures $x(t + 1)$ incurred in that period stems from the production $y(t)$ generated in the previous period: $x(t + 1) \leqq y(t)$, $t = 0, 1, \ldots, t_1 - 1$. It follows from the condition (**G3**) that:

$$(y(t), y(t + 1)) \in Z(t + 1), \quad t = 0, 1, \ldots, t_1 - 1. \tag{9}$$

The initial state of the economy (initial production vector) is fixed at period $t = 0$:

$$y(0) = y^0 \geq 0. \tag{10}$$

Every sequence of production vectors $\left\{y(t)\right\}_{t=0}^{t_1}$ which satisfies conditions (9)-(10) is called a (y^0, t_1) – feasible growth process (production trajectory) in the non-stationary Gale economy with a multiline turnpike and varying technology. With all the accepted assumptions such processes exist $\forall y^0 \geq 0\ \forall t_1 < +\infty$. It is assumed that a sequence of optimal production sets $\left\{Z_{opt}(t)\right\}_{t=0}^{t_1}$ satisfies the following conditions for $\overline{t}\geq 0$:

[13] This condition excludes (from the geometric point of view) a specific shape of the production space $Z(t)$ which otherwise for $t \to +\infty$ could "expand" in the neighbourhood of the bundle of optimal processes which generate the multiline turnpike.

(G8) $$\exists\{\overline{x}(t),\overline{y}(t)\}_{t=\overline{t}}^{t_1}\ \forall t\in\{\overline{t},\overline{t}+1,\ldots,t_1\}\big((\overline{x}(t),\overline{y}(t))\in Z_{opt}(t)\big)\wedge$$
$$\wedge\forall t\in\{\overline{t},\overline{t}+1,\ldots,t_1-1\}\big(\overline{x}(t+1)=\overline{y}(t)\big).$$

☐ **Lemma 2.** In the Gale economy with conditions **(G1)**-**(G8)** a sequence of production vectors $\{\overline{y}(t)\}_{t=\overline{t}}^{t_1}$, distinguished in **(G8)**, satisfies:

$$\big(\overline{y}(t),\overline{y}(t+1)\big)\in Z(t+1),\ t=\overline{t},\overline{t}+1,\ldots,t_1-1,\overline{y}(t)=$$
$$=\left(\prod_{\theta=\overline{t}+1}^{t}\alpha_{M,\theta}\right)\overline{y}(\overline{t}),t=\overline{t}+1,\ldots,t_1. \tag{11}$$

Proof. Since $\big(\overline{x}(t),\overline{y}(t)\big)\in Z_{opt}(t)$, then $\alpha_{M,t}\overline{x}(t)\leqq\overline{y}(t)$ and (under **(G3)**)

$$\big(\overline{x}(t),\alpha_{M,t}\overline{x}(t)\big)\in Z_{opt}(t),\quad t=\overline{t},\overline{t}+1,\ldots,t_1.$$

According to **(G8)** the outputs match the inputs $\overline{x}(t+1)=\overline{y}(t)$ hence $\big(\overline{y}(t),\overline{y}(t+1)\big)\in Z_{opt}(t+1)$, where:

$$\overline{y}(t+1)=\alpha_{M,t+1}\overline{y}(t),\quad t=\overline{t},\overline{t}+1,\ldots,t_1.$$

The latter condition is equivalent to (11). ■

A sequence of production vectors (11) has a constant structure:

$$\forall t\in\{\overline{t},\overline{t}+1,\ldots,t_1\}\left(\frac{\overline{y}(t)}{\|\overline{y}(t)\|}=\frac{\overline{y}(\overline{t})}{\|\overline{y}(\overline{t})\|}=\overline{s}\in S(\overline{t})\right),$$

and defines a $(\overline{t},t_1)$ – stationary growth process. All production vectors (elements of the sequence) belong to the turnpike (to a single von Neumann ray) $N_{\overline{s}}^{\overline{t}}$:

$$\overline{y}(t)\in N_{\overline{s}}^{\overline{t}}=\{\{\lambda\overline{s}\mid\lambda>0\}\}\in\mathbb{N}^t,\quad t=\overline{t},\overline{t}+1,\ldots,t_1.$$

If the trajectory $\{\overline{y}(t)\}_{t=\overline{t}}^{t_1}$ is a stationary process of type (11), then every trajectory $\{\lambda\overline{y}(t)\}_{t=\overline{t}}^{t_1}$, $\lambda>0$ is of the same type. All those vectors belong to the turnpike $N_{\overline{s}}^{\overline{t}}$. This particular, single turnpike which starts at period $\overline{t}$ and extends to the horizon T is called the peak turnpike (peak von Neumann ray) in the non-stationary Gale economy. On the peak turnpike $N_{\overline{s}}^{\overline{t}}$ the economy (starting at period $\overline{t}$) reaches the highest growth rate and maximal technological and economic production efficiency.[14]

[14] Among other single turnpikes which determine a multilane turnpike $\mathbb{N}^t$, they could exist in the horizon T as well as in shorter periods. In none, except for the peak turnpike $N_{\overline{s}}^{\overline{t}}$, does there exist a guaranteed and permanent, maximal growth rate or highest possible production efficiency.

3. Optimal growth processes. Turnpike effect

The solution of the following target growth problem is of interest[15]:

$$\max\left\langle \overline{p}(t_1), y(t_1)\right\rangle \quad \text{under conditions (9)-(10)} \quad (\text{given } y^0). \tag{12}$$

This problem has a solution under assumptions **(G1)-(G8)** denoted by $\{y^*(t)\}_{t=0}^{t_1}$ and called a $\left(y^0, t_1, \overline{p}(t_1)\right)$ – optimal growth process. Its relationships with a multilane turnpike in the special case, when the $\left(y^0, t_1, \overline{p}(t_1)\right)$ – optimal growth process at period $\check{t} \in \left\{\overline{t}, \overline{t}+1, \ldots, t_1-1\right\}$ reaches the peak turnpike $N_{\overline{s}}^{\overline{t}}$ is described in the following theorem.

■ **Theorem 3.** The horizon $T = \{0, 1, \ldots, t_1\}$ ($t_1 < +\infty$) and a number $\varepsilon > 0$ are fixed. Let $\{y^*(t)\}_{t=0}^{t_1}$ denote an $\left(y^0, t_1, \overline{p}(t_1)\right)$ – optimal growth process which at period $\check{t} \in \left\{\overline{t}, \overline{t}+1, \ldots, t_1-1\right\}$ satisfy a condition:

$$y^*(\check{t}) \in N_{\overline{s}}^{\overline{t}}.$$

Under the conditions **(G1)-(G8)**, if the von Neumann prices are non-increasing at the periods $t = \overline{t}, \overline{t}+1, \ldots, t_1$[16]:

$$\overline{p}(t+1) \leqq \overline{p}(t), \quad t = \overline{t}, \overline{t}+1, \ldots, t_1-1, \tag{13}$$

and satisfy the condition[17]:

$$\frac{\left\langle \overline{p}(\check{t}), \overline{s}\right\rangle}{\left\langle \overline{p}(t_1), \overline{s}\right\rangle} < \frac{1}{1-\nu_\varepsilon}, \tag{14}$$

[15] Task of maximization of the value of production measured in von Neumann prices at the target period t_1 of the horizon T.

[16] Von Neumann prices are determined up to structure so to fulfill that condition it is enough to guarantee that at the periods $t = \overline{t}, \overline{t}+1, \ldots, t_1$ the prices $\overline{p}$(t) are positive. It is easy to show that under the conditions **(G1)-(G8)** this is true. Suppose not, then $\exists t \exists i \in \{1, \ldots, n\}\left(\overline{p}_i(t) = 0\right)$. Hence $\exists\left(\overline{x}(t), \overline{y}(t)\right) \in Z_{opt}(t) \subset Z(t)\setminus\{0\}\left(\overline{y}(t) = \alpha_{M,t}\overline{x}(t)\right)$ and $\beta\left(\overline{x}(t), \overline{y}(t), \overline{p}(t)\right) = \alpha_{M,t} > 0$. Let $\tilde{x}(t) = \overline{x}(t) + e^i$, where $e^i = (0, \ldots, 1, \ldots, 0)$ is an n-dimensional vector with a 1 on the i-th position. Thus $\left(\tilde{x}(t), \overline{y}(t)\right) \in Z(t)\setminus\{0\}$ and $\alpha\left(\tilde{x}(t), \overline{y}(t)\right) < \alpha_{M,t}$. Hence under (G6)):

$$\beta\left(\tilde{x}(t), \overline{y}(t), \overline{p}(t)\right) < \alpha_{M,t}$$

On the other hand, the definition of $\beta(\cdot)$ (since $\overline{p}_i(t) = 0$) implies:

$$\beta\left(\tilde{x}(t), \overline{y}(t), \overline{p}(t)\right) = \frac{\left\langle \overline{p}(t), \overline{y}(t)\right\rangle}{\left\langle \overline{p}(t), \tilde{x}(t)\right\rangle} = \frac{\left\langle \overline{p}(t), \overline{y}(t)\right\rangle}{\left\langle \overline{p}(t), \overline{x}(t)\right\rangle} = \alpha_{M,t}$$

The contradiction proves that $\overline{p}_i(t) > 0$.

[17] Starting with the period until the end of horizon the prices do not change rapidly.

then:

$$\forall t \in \left\{\check{t}+1, \ldots, t_1 - 1\right\}\left(d\left(y^*(t), \mathbb{N}^{t+1}\right) < \varepsilon\right).$$

Proof. Every $\left(y^0, t_1, \overline{p}(t_1)\right)$ – optimal growth process satisfies the condition (4), hence:

$$\left\langle \overline{p}(t+1), y^*(t+1)\right\rangle \le \alpha_{M,t+1}\left\langle \overline{p}(t+1), y^*(t)\right\rangle, \quad t = 0, 1, \ldots, t_1 - 1,$$

In particular for $t = t_1 - 1$ there is:

$$\left\langle \overline{p}(t_1), y^*(t_1)\right\rangle \le \alpha_{M,t_1}\left\langle \overline{p}(t_1), y^*(t_1 - 1)\right\rangle,$$

and further because of the monotonicity of the prices (10),

$$\left\langle \overline{p}(t_1), y^*(t_1)\right\rangle \le \alpha_{M,t_1}\left\langle \overline{p}(t_1), y^*(t_1-1)\right\rangle \le \alpha_{M,t_1}\left\langle \overline{p}(t_1-1), y^*(t_1-1)\right\rangle \le$$
$$\le \alpha_{M,t_1-1}\alpha_{M,t_1}\left\langle \overline{p}(t_1-1), y^*(t_1-2)\right\rangle.$$

That leads to the inequality:

$$\left\langle \overline{p}(t_1), y^*(t_1)\right\rangle \le \left(\prod_{t=\hat{t}+1}^{t_1}\alpha_{M,t}\right)\left\langle \overline{p}(\check{t}), y^*(\check{t})\right\rangle = \sigma\left(\prod_{t=\hat{t}+1}^{t_1}\alpha_{M,t}\right)\left\langle \overline{p}(\check{t}), \overline{s}\right\rangle, \qquad (15)$$

where $\sigma = \left\|y^*(\check{t})\right\| > 0$; $\overline{s}$ is a vector of production structure on the peak turnpike $N_{\overline{s}}^{\overline{t}}$, which is reached at period $t = \check{t}$ by the $\left(y^0, t_1, \overline{p}(t_1)\right)$ optimal growth process.

Assume that at certain period $t' \in \{\check{t} + 1, \ldots, t_1 - 1\}$:

$$d\left(y^*(t'), \mathbb{N}^{t'+1}\right) \ge \varepsilon.$$

Then due to lemma 1,

$$\left\langle \overline{p}\,(t'+1), y^*(t'+1)\right\rangle \le \left(\alpha_{M,t'+1} - \delta_{\varepsilon,t'+1}\right)\left\langle \overline{p}\,(t'+1), y^*(t')\right\rangle, \qquad (16)$$

where $\delta_{\varepsilon,t'+1} \in \left(0, \alpha_{M,t'+1}\right)$. Jointly (15), (16) provide an upper boundary for the production generated in the optimal process at the final period of the horizon T:

$$\left\langle \overline{p}\,(t_1), y^*(t_1)\right\rangle \le \sigma\left(\alpha_{M,t'+1} - \delta_{\varepsilon,t'+1}\right)\left(\prod_{\substack{t=\check{t}+1 \\ t \ne t'+1}}^{t_1}\alpha_{M,t}\right)\left\langle \overline{p}(\check{t}), \overline{s}\right\rangle. \qquad (17)$$

Since the $\left(y^0, t_1, \overline{p}(t_1)\right)$ – optimal growth process at period $t = \check{t}$ reaches the peak turnpike $y^*(\check{t}) \in N_{\overline{s}}^{\overline{t}}$, the following sequence of production vectors $\left\{\tilde{y}(t)\right\}_{t=0}^{t_1}$:

$$\tilde{y}(t) = \begin{cases} y^*(t), & t = 0, 1, \ldots, \check{t} \\ \sigma\left(\prod_{\theta=\check{t}+1}^{t} \alpha_{M.\theta}\right)\overline{s}, & t = \check{t}+1, \ldots, t_1 \end{cases}$$

determines an (y^0, t_1) – an feasible process.[18] The inequality:

$$\left\langle p(t_1), y^*(t_1)\right\rangle \geq \left\langle p(t_1), \tilde{y}(t_1)\right\rangle$$

and (17) imply:

$$\sigma\left(\alpha_{M,t'+1} - \delta_{\varepsilon,t'+1}\right)\left(\prod_{\substack{t=\check{t}+1 \\ t\neq t'+1}}^{t_1} \alpha_{M,t}\right)\left\langle \overline{p}(\check{t}), \overline{s}\right\rangle \geq \sigma\left(\prod_{t=\check{t}+1}^{t_1} \alpha_{M,t}\right)\left\langle \overline{p}(t_1), \overline{s}\right\rangle > 0.$$

That proves (under **(G7),** (13), (14)) the inequalities:

$$\frac{1}{1-\nu_\varepsilon} \leq \frac{\alpha_{M,t'+1}}{\alpha_{M,t'+1} - \delta_{\varepsilon,t'+1}} \leq \frac{\left\langle \overline{p}(\check{t}), \overline{s}\right\rangle}{\left\langle \overline{p}(t_1), \overline{s}\right\rangle} < \frac{1}{1-\nu_\varepsilon}.$$

The contradiction proves the claim. ■

Conclusions

The theorem 3 remains valid when one replaces problem (12) with the following

$$\begin{gathered} \max u\left(y(t_1)\right) \\ \text{under conditions (9)-(10)} \\ (\text{given } y^0). \end{gathered} \tag{12'}$$

In problem (12′) the production utility of the output generated at the final period of the horizon T is maximised, where u: $R_+^n \rightarrow R_+^1$ is a continuous, concave, positively homogenous of degree 1 utility function which satisfies the following condition[19]:

[18] It is obtained by fusing the initial part (for $t = 0, 1, \ldots, \check{t}$) of the optimal production trajectory $\left\{y^*(t)\right\}_{t=0}^{t_1}$ with a $(\check{t}, t_1)$- stationary trajectory of type (11) (for $t = \check{t} + 1, \ldots, t_1$).

[19] The condition says that the utility function can be approximated from above by a linear form with the coefficient vector, tangent to the utility function along the top turnpike.

(G9) $\exists a > 0 \, \forall s \in S_+^n(1) \left(u(s) \le a \left\langle \overline{p}(t_1), s \right\rangle \right)$ and (on the peak turnpike $N_{\overline{s}}^{\overline{t}}$):

$$u(\overline{s}) = a \left\langle \overline{p}(\check{t}), \overline{s} \right\rangle.$$

It remains an interesting research objective to study the stability of optimal growth processes in the non-stationary Gale economy with a multilane turnpike without condition **(G8)**. The case of a non-stationary Gale economy with a single turnpike was discussed partially in the papers Panek (2015a, b).

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