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Three theories of equilibrium in mathematical economics

An attempt at a synthesis*

1. Introduction

The concept of equilibrium, as used in economic sciences, is not without ambiguities. Mathematical economics is heavily influenced by the Walrasian notion of the competitive equilibrium which is a state of an economy, described in terms of the size and structure of production, production factors and price levels, wherein the demand for products and production factors equals their supply. It is assumed that production and exchange processes are fully governed by market mechanisms. Any "external" interference, e.g. by governmental attempts to regulate prices, is out of the question as such interference would disrupt the balance between market forces.

In technical science an equivalent of the Walrasian notion of the competitive equilibrium is a state of a (static) equilibrium attained by an autonomous object that follows its own "laws of motion" and that is not affected by any external forces. The equilibrium is a position that such an object, e.g. a pendulum, will reach by itself without operation of any external forces and will maintain until disrupted by an external impulse.

A completely different is the idea of so called von Neumann equilibrium. Here, an economy will maintain its equilibrium provided that it is capable of increasing production of each good at the same constant rate and that its technological development is synchronized with economic growth described in value terms. The concept of the economic equilibrium as proposed by J. von Neumann is flawed in that it restricts equilibria to the realm of production. In the von Neumann model real consumption problems are shadowed by the essential question of ensuring a steady growth of production.

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An intermediate concept falling in between the notions of Walras and von Neumann is that of the neoclassical equilibrium which allows for a balanced growth of all of such fundamental economic variables as capital, production (revenues), consumption and investment.

These three currents in mathematical economics are used here as a backdrop for reflections on the nature of the economic equilibrium in an attempt to establish:

- what qualities are shared by all of the above equilibrium theories,
- whether there is an "absolute" economic equilibrium, and
- whether the theory of economic equilibria can be extended beyond the magic circle of stationary states.

In the conclusions to the discussion, a presentation is provided of the concept of a dynamic σ- equilibrium formulated on the basis of the systems theory. The concept accounts for the specific nature and relativity of the notion of equilibrium as applied to complex socio-economic systems.

2. The competitive equilibrium: the Arrow-Debreu-McKenzie model

In this section a model of a market economy is presented, in which business entities are naturally opposed to one another as they compete with one another in pursuing their individual business strategies. Amidst the resulting clashes between the (typically) opposed interests of a large number of economic players, the economic agents are unable to exert a direct impact on price levels. All they can do is to passively accept the existing system and work with their own best offers to buy or sell goods. Not all price configurations will accommodate their individual preferences. Formulated over a century ago by L. Walras, the notion of competitive equilibrium rests on the hypothesis that certain configurations of prices allow for the individual goals of producers and consumers to mutually agree and satisfy their needs.

The Arrow-Debreu-McKenzie (A-D-McK) model, presented further in this Section, deserves particular attention for two reasons:

- firstly, it is one of the most general models of a competitive economy; most other models are either its specific case versions or are its more or less elaborated extensions,
- secondly, the model provides a clear and straightforward mathematical structure, a feature that four decades ago allowed its authors to prove a theorem on existence of the state of competitive equilibrium.

Assume that an economy comprises (uses up or manufactures) *n* goods. We will not specify what proportion of these are consumer goods, which are production factors and which can be either (such as a computer that, depending on its use, may serve as either a production factor or a consumer item). Such a generalization is common in contemporary mathematical economics and has the advantage of allowing for the application of many fundamental theorems pertaining to topological structures of compact convex sets.

Let time t be a continuous variable that assumes values within interval T = [0, +∞). Set T is called the time horizon (here unbounded) of the economy. The moment t = 0 is the starting point of horizon T. By $p(t)$ we denote an n-dimensional (row) vector of prices of goods at time t. Symbols m and l denote respectively, the number of producers and the number of consumers.

Further, we will assume that consumers are characterized by:

– preference fields

$$\left(X^k, u^k\right)_{k=1}^{l}, \quad X^k \subset R^n,$$

where the set X^k is a space of goods of the *k-th* consumer with a metric generated by the norm on space R^n, and u^k is an individual utility function of the *k-th* consumer,

– initial stocks of goods

$$a^k = \left(a_1^k, ..., a_n^k\right)^T \geq 0,$$

where $k = 1, 2, \ldots, l$ and T is the transposition mark.

Elements of space X^k are called goods baskets. They are n-dimensional (column) vectors with positive components pointing to goods whose purchase is sought by the *k-th* consumer and negative components, pointing to goods that the consumer is seeking to sell (such as labor).

The production capacities of producers are represented by their production spaces,

$$\left(Y^j\right)_{j=1}^{m}, \quad Y^j \subset R^n.$$

The elements of Y^j describe feasible production processes. They are n-dimensional vectors with components that may be positive (positive final production of goods), negative (negative final production of goods) or zero (simple reproduction of goods).

The *j-th* producer employs at time t the criterion of maximization of profit at current prices $p(t)$ and selects the (column) production vector $y^j(t) = \left(y_1^j(t), ..., y_n^j(t)\right)^T$ that satisfies the following condition:

$$y^j(t) = g^j\left(p(t)\right) = \arg\max_{y \in Y^j} \left\langle p(t), y \right\rangle \qquad j = 1, \ldots, m. \qquad (1)$$

The income of the *k-th* consumer at time t, denoted by $I^k\left(p(t), y^1(t), ..., y^m(t)\right)$, is partially derived from the sales of initial stocks $a^k \geq 0$ and partially from his shares in the profits of producers:

$$I^k\left(p(t), y^1(t), \ldots, y^m(t)\right) = \left\langle p(t), a^k \right\rangle + \sum_{j=1}^{m} \alpha_{kj} \left\langle p(t), y^j(t) \right\rangle \quad k = 1, \ldots, l, \quad (2)$$

where $\alpha_{kj} \geq 0$ denotes the share of the *k-th* consumer in the profit of the *j-th* producer, $\sum_{k=1}^{l} \alpha_{kj} = 1 \quad j = 1, \ldots, m$.

It follows from (1) and (2) that:

$$I^k\left(p(t), y^1(t), \ldots, y^m(t)\right) = I^k\left(p(t), g^1(p(t)), \ldots, g^m(p(t))\right) = \\ = \left\langle p(t), a^k \right\rangle + \sum_{j=1}^{m} \alpha_{kj} \xi^j(p(t)) = \varphi^k(p(t)), \qquad k = 1, \ldots, l,$$

where $$\xi^j(p(t)) = \max_{y \in Y^j} \left\langle p(t), y \right\rangle = \left\langle p(t), g^j(p(t)) \right\rangle \qquad j = 1, \ldots, m.$$

Above equations imply that ultimately the profits of produces $\xi^j(p(t))$ and incomes of consumers $\varphi^k(p(t))$ depend exclusively on the price vector $p(t)$.

The *k-th* consumer is restricted in his choice of goods to these baskets whose value does not exceed $\varphi^k(p(t))$. As a result, at time t, the consumer maximizes utility selecting basket of goods $x^k(t) = \left(x_1^k(t), \ldots, x_n^k(t)\right)^T$ that satisfies the condition:

$$x^k(t) = f^k(p(t)) = \arg\max_{\substack{\langle p(t), x \rangle \leq \varphi^k(p(t)) \\ x \in X^k}} u^k(x) \qquad k = 1, \ldots, l. \qquad (3)$$

We shall asume that goods spaces, production spaces and initial inventory vectors as well as the utility functions satisfy the following conditions:

(i) the sets X^k are closed in R^n, convex, bounded below and such that when selecting any basket $x \in X^k$, the consumer may always point to a basket $x' \in X^k$ of higher utility that satisfies the condition $u^k(x') > u^k(x)$;

(2i) the utility functions u^k are continuously differentiable, increasing and concave on X^k, $k = 1, \ldots, l$;

(3i) the sets Y^j are closed in R^n and convex; moreover $Y^j \cap R_+^n = \{0\}$, $Y^j \cap \left(-Y^j\right)$ and such that if $y \in Y^j$, then $y' \in Y^j$ for any vector $y' \leq y$;

(4i) int $\left(Y + \left\{a^k\right\}\right) \cap X^k \neq \varnothing$, $\qquad k = 1, \ldots, l$

where $Y = \sum_{j=1}^{m} Y^j = \left\{ y \,\middle|\, y = \sum_{j=1}^{m} y^j;\ y^j \in Y^j \right\}$ is the global production space.

The boundedness from below of sets X^k mentioned in (i) actually refers to negative coordinates describing available production factors that are shown in baskets $x^k \in X^k$ with the (–) sign and that are inherently limited.

Condition (2i) imposed on the consumer utility function is standard. Condition (3i) expresses three postulates:

– the first of them is known as the *no cornucopia postulate* (no production without inputs),
– the second postulate establishes a technological regime wherein production processes cannot be "reversed" (if input *a* allows for the production of output *b≠a*, then the process cannot be reversed, i.e. products *b* cannot be turned back into the initial collection of goods *a*),
– the third postulate points to the possibility of waste (if it is possible to produce the goods vector *y*, then it is also possible to produce fewer such goods).

In accordance with condition (4i), the market holds something of interest for every consumer and the range of goods available is so wide that consumers have the choice not only of tapping into a single basket of goods but also of other baskets comprising similar assortments.

While every producer tries to ensure the maximum profit at given prices, every consumer makes an effort to select what he believes to be the best basket of goods he can afford given his particular income. In an competitive equilibrium, all of these mutually opposed activities take place without upsetting the global balance between supply and demand and without distorting individual equilibria between consumers' incomes and expenditures.

Definition 1. The vector system

$$\left(\bar{x}^1, ..., \bar{x}^l, \bar{y}^1, ..., \bar{y}^m, \bar{p}\right)$$

with the price vector $\bar{p} \gneqq 0$ constitutes an equlibrium of the A-D-McK economy whenever it satisfies the following conditions:

(I) every producer maximizes his profit at equilibrium prices:

$$\left\langle \bar{p}, \bar{y}^j \right\rangle = \xi^j\left(\bar{p}\right) = \max_{y \in Y^j} \left\langle \bar{p}, y \right\rangle \qquad j = 1, \ldots, m.$$

(II) the expenditure of each individual consumer does not exceed his income:

$$\bar{x}^k \in X^k \quad \text{and} \quad \left\langle \bar{p}, \bar{x}^k \right\rangle \le \varphi^k(\bar{p}),$$

where $\varphi^k(\bar{p}) = \langle \bar{p}, a^k \rangle + \sum_{j=1}^{m} \alpha_{kj} \xi^j(\bar{p}),$ $k = 1, \ldots, l.$

(III) every consumer maximizes utility subject to his budget constraint at equlibrium prices:

$$u^k(\bar{x}^k) = \max u^k(x) \qquad k = 1, \ldots, l.$$
$$\langle \bar{p}, x \rangle \leq \varphi^k(\bar{p})$$
$$x \in X^k$$

(IV) the global demand for each good does not exceed its supply:

$$\sum_{k=1}^{l} \bar{x}^k \leq \sum_{k=1}^{l} a^k + \sum_{j=1}^{m} \bar{y}^j$$

Above definition is one of the most general definitions of a competitive equilibrium in mathematical economics. The proof of the existence of a competitive equilibrium under assumptions (i) – (4i) can be found e.g. in (Panek 1993, 2000). In addition, extensive literature is available providing proofs of the existence of competitive equilibria in other models which represent more or less close approximations of the A-D-McK model (cf. e.g. Allingham 1975, Intriligator 1971, Mas-Coell 1985, Mukherji 1990, Nikaido 1968).

The very existence of a competitive equilibrium would not attract any special attention among economists were it not for the strong belief of free market advocates that the market adjustments lead the economy towards equilibrium, which in fact is a belief in the inherent stability (local or global) of economies.

In order to explore the nature of such a stability, we define the function of surplus demand at time t as the difference between global demand $\sum_{k=1}^{l} f^k(p(t))$ and global supply $\sum_{k=1}^{l} a^k + \sum_{j=1}^{m} g^j(p(t))$:

$$F(p(t)) = \sum_{k=1}^{l} f^k(p(t)) - \sum_{k=1}^{l} a^k - \sum_{j=1}^{m} g^j(p(t))$$

where in accordance with (1) and (3):

$$f^k(p(t)) = x^k(t) \quad \text{and} \quad g^j(p(t)) = y^j(t).$$

Further we shall need two additional assumptions:

(5i) the function of surplus demand $F(p)$ is continuous and differentiable everywhere on its domain except zero and satisfies the condition:

$$p_i = 0 \Rightarrow F_i(p) > 0$$

(the demand for goods offered free of charge always exceeds their supply),

(6i) the inequality

$$\lambda \frac{\partial F}{\partial p} \lambda^T < 0$$

is satisfied for every vector $\lambda = (\lambda_1, ..., \lambda_n)$, that contains at least one positive and one negative element (this is the standard condition for the so-called normal markets in which an increase in the price of *i-th* good reasults in stronger drop of demand for that good than any other goods).

Note that the surplus demand does not depend on the level of prices but rather on their structure:

$$F(\lambda p) = F(p) \qquad \text{for } \lambda > 0 .$$

Consequently, equilibrium prices are also defined with accuracy up to the structure, i.e. if the system of vectors for $(\bar{x}^1, ..., \bar{x}^l, \bar{y}^1, ..., \bar{y}^m, \bar{p})$ describes an economy in a state of equilibrium, then the system $(\bar{x}^1, ..., \bar{x}^l, \bar{y}^1, ..., \bar{y}^m, p')$ with any price vector *p*' on the ray

$$P = \{ \lambda\bar{p} \mid \lambda > 0 \}$$

also describes an equilibrium.

Assume that prices change according to the standard equation:

$$\frac{d}{dt} p(t) = \sigma F(p(t)) \qquad (4)$$

where ($\sigma > 0$) and at the initial point of time $t = 0$ prices are given:

$$p(0) = p^0 > 0 \qquad (5)$$

A non-negative solution to the set of equations (4) with the initial condition (5) is called the feasible trajectory of prices in the A-D-McK economy. The functions $y^j(t)$, $x^k(t)$ that correspond thereto in accordance with (1) and (3) are called (respectively) the feasible production trajectory (of the *j-th* producer) and the feasible demand trajectory (of the *k-th* consumer); $j = 1, \dots, m$; $k = 1, \dots, l$.

Assume that vectors $(\bar{x}^1, ..., \bar{x}^l, \bar{y}^1, ..., \bar{y}^m, \bar{p})$ describe a competitive equilibrium in the A-D-McK economy.

Definition 2. The economy is globally asymptotically stable if, for any initial price vector $p^0 > 0$, every feasible price trajectory $\{p(t)\}_{t=0}^{\infty}$ is convergent to a certain vector of equilibrium prices:

$$\lim_t p(t) = \bar{p} \in P .$$

Under assumptions (i)-(6i), there is such a price vector $\bar{p}$, determined up to the structure, that $F(\bar{p}) = 0$ (global demand is equal to global supply) and the A-D-McK economy is globally asymptotically stable (Panek 1997, 2000). Moreover:

$$y^j(t) = g^j(p(t)) \xrightarrow[t]{} g^j(\bar{p}) = \bar{y}^j \qquad j = 1, \ldots, m,$$

$$x^k(t) = f^k(p(t)) \xrightarrow[t]{} f^k(\bar{p}) = \bar{x}^j \qquad k = 1, \ldots, l,$$

i.e. as time passes, not only are price trajectories convergent to the equilibrium price vector but, in addition, production and demand trajectories are convergent to the production and demand vectors in a state of equilibrium. The price trajectory $p(t)$ is situated on the surface of an n-dimensional sphere with the radius $r = \sqrt{\langle p^0, p^0 \rangle} > 0$. Its convergence to the ray P of equilibrium prices is illustrated in Fig. 1.

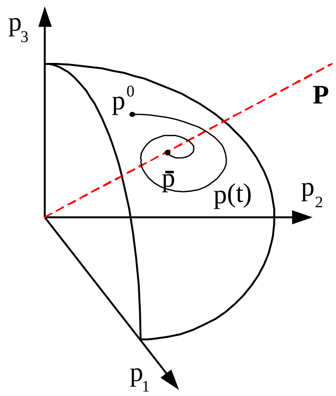

Fig. 1. Illustration of the convergence of price trajectory *p*(*t*) to the equilibrium price ray in R^3

3. The von Neumann equilibrium

It is difficult to overestimate the contribution of J. von Neumann to mathematics, economics and even computer science (binary systems). In a strange coincidence, it was not until after World War II (1946) that his equilibrium model, first published in 1937, was noticed and attracted economists' attention.

In the von Neumann economy a finite number of goods n are used and produced with the use of a finite number of production processes m called base technological processes. The number of base technological processes may be larger than, smaller than or equal to the quantity of goods.

As in the A-D-McK model, the terms goods and process are used in a broad sense. Thus, goods include land, materials, fuels, energy, semi-finished products, finished products, production facilities depreciated to varying degrees, labor having varying qualifications, etc. Technological processes are identified with activities whose effect is to turn one type of goods into another. Such processes may simultaneously refer to activities in the area of production, investment, consumption, transportation, warehousing, education, etc.

Let $a_j = (a_{j1}, ..., a_{jn})$ denote a (row) vector of inputs and $b_j = (b_{j1},...,b_{jn})$ a (row) vector of outputs of the *j-th* base technological process carried out with the unit intensity, $j = 1, \ldots, m$. Let A and B denote rectangular $m \times n$ matrices:

$$A = \begin{pmatrix} a_{11}, \ldots, a_{1n} \\ a_{21}, \ldots, a_{2n} \\ \vdots \\ a_{m1}, \ldots, a_{mn} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11}, \ldots, b_{1n} \\ b_{21}, \ldots, b_{2n} \\ \vdots \\ b_{m1}, \ldots, b_{mn} \end{pmatrix}.$$

The non-negative matrix A is called the input matrix , whereas the non-negative matrix B is the output matrix. We assume that:

(i) each row of matrix A contains at least one positive element,

(2i) each column of matrix B contains at least one positive element.

It follows from assumption (i) that at least one type of goods is used in the base technological process (the *no cornucopia* postulate described in the previous Section). Under (2i), every type of goods is made with the use of at least one base technological process.

The model is linear in the sense that any non-negative linear combination of base technological processes is a feasible production process, i.e. for any (row) vector $\vartheta = (\vartheta_1, ..., \vartheta_m) \geq 0$, inputs ϑA can be used to produce outputs ϑB. Vector ϑ describes intensities at which the base technological processes are used. In the von Neumann model, the set

$$Q = \left\{ q \mid q = \vartheta B - \vartheta A; \quad \vartheta \geq 0 \right\}$$

is a global production space (we saw such a space in the previous point under assumption (4i)).

Consider an intensity vector $\vartheta \geq 0$ and process (ϑA, ϑB). The number,

$$\alpha_i(\vartheta) = \begin{cases} \dfrac{(\vartheta B)_i}{(\vartheta A)_i}, & \text{when} \quad (\vartheta A)_i > 0 \\ +\infty, & \text{when } (\vartheta A)_i = 0 \text{ and } (\vartheta B)_i > 0 \\ \text{an indeterminate form,} & \text{when } (\vartheta A)_i = (\vartheta B)_i = 0 \end{cases}$$

is called the technological effectiveness of production of the *i-th* good in process (ϑA, ϑB). The number

$$\alpha(\vartheta) = \min_i \alpha_i(\vartheta)$$

represents, in turn, the technological effectiveness of the whole process (ϑA, ϑB). Finally, the number

$$\alpha_M = \max_{\vartheta \gneq 0} \alpha(\vartheta)$$

represents the optimal technological effectiveness of production. Vector $\overline{\vartheta} \gneq 0$, for which $\alpha_M = \alpha(\overline{\vartheta})$, is an optimal intensity vector, whereas vectors $\overline{x} = \overline{\vartheta} A$ and $\overline{y} = \overline{\vartheta} B$ represent the optimal input and output vectors.

Under assumptions (i) and (2i), there is an optimal intensity vector in the von Neumann economy defined with accuracy up to the structure (by the same token, there exist optimal input and output vectors defined with accuracy up to the structure (Panek 1997, 2000).

Let $p \gneq 0$ denote an n-dimensional (column) price vector. The number

$$\beta(\vartheta, p) = \begin{cases} \dfrac{\vartheta Bp}{\vartheta Ap}, & \text{when} \quad \vartheta Ap > 0 \\ +\infty, & \text{when } \vartheta Ap = 0 \text{ and } \vartheta Bp > 0 \\ \text{an indeterminate form,} & \text{when } \vartheta Ap = \vartheta Bp = 0 \end{cases}$$

represents the economic effectiveness of process (ϑA, ϑB) at prices p.

Definition 3. The intensity vector $\overline{\vartheta} \gneq 0$, the price vector $\overline{p} \gneq 0$ and number $\alpha > 0$ describe an equlibrium in the von Neumann economy if they satisfy the following conditions:

(I) $\alpha \overline{\vartheta} A \le \overline{\vartheta} B$

(II) $B\overline{p} \le \alpha A \overline{p}$

(III) $\overline{\vartheta} B \overline{p} > 0$

Vector $\bar{\vartheta}$ is the equlibrium intensity vector describing intensities at which the base technological processes are used in equilibrium. Vector $\bar{p}$ in turn, is the equilibrium price vector.

Prices and the production intensities in equilibrium are defined with accuracy up to the structure.

It can be shown that the number α represents the technological effectiveness of the equilibrium process $(\bar{\vartheta}A, \bar{\vartheta}B)$:

$$\alpha = \alpha(\bar{\vartheta}).$$

Moreover

$$\alpha(\bar{\vartheta}) = \beta(\bar{\vartheta}, \bar{p}) \geq \beta(\vartheta, \bar{p})$$

for each intensity vector $\vartheta \geq 0$ for which formula $\beta(\vartheta, \bar{p})$ is defined.

In other words, in the von Neumann equilibrium, the technological effectiveness of production equals the economic effectiveness and represents the maximum effectiveness the economy can achieve at equilibrium prices.

It can be demonstrated that, under assumptions (i) and (2i), there is a state of equilibrium characterized by the optimal technological effectiveness $\alpha = \alpha_M$ and that the number of von Neumann equilibria at various levels of technological effectiveness $\alpha \leq \alpha_M$ does not exceed min$\{m, n\}$, (Czeremnych 1982).

The particular state of equilibrium $(\alpha_M, \bar{\vartheta}, \bar{p})$ is called the optimal equilibrium in the von Neumann economy.

Assume now that time changes descreetly and that the (bounded) horizon of the economy $T = \{0, 1, \ldots, t_1\}$. Let $\vartheta(t)$ be an m-dimensional row vector of the intensities of the base technological processes in period t.

The von Neumann economy is closed in the sense that the sole source of inputs in the subsequent period $t + 1$ is the production (output) from the previous period t. Formally, that means that the follwoing inequalities are satisfied:

$$\begin{aligned} &\vartheta(t+1)A \leq \vartheta(t)B \qquad t = 0, 1, \ldots, t_1 - 1, \\ &\vartheta(t) \geq 0 \qquad t = 0, 1, \ldots, t_1, \end{aligned} \tag{6}$$

or alternatively:

$$\Delta\vartheta(t) \in \Phi(\vartheta(t)), \qquad t = 0, 1, \ldots, t_1 - 1 \tag{6a}$$

where Φ is a multifunction:

$$\Phi(\vartheta(t)) = \{ z \mid z = \vartheta - \vartheta(t), \quad \vartheta A \leq \vartheta(t)B, \quad \vartheta \geq 0 \}.$$

Let $\vartheta^0 \gneqq 0$ represents the intensity at which base technological processes are employed in the initial period ($t = 0$).

$$\vartheta(0) = \vartheta^0 \qquad (7)$$

The sequence of vectors $\{\vartheta(t)\}_{t=0}^{t_1}$ that satisfy conditions (6) and (7) is called the feasible trajectory of intensity in the von Neumann model. The corresponding sequences $\{x(t)\}_{t=0}^{t_1}$, $\{y(t)\}_{t=0}^{t_1}$, where

$$x(t) = \vartheta(t)A, \quad y(t) = \vartheta(t)B \qquad (8)$$

are called (respectively) feasible input and output trajectories.

From all feasible intensity trajectories, particular importance is attributed to trajectories assuming the form:

$$\vartheta(t) = \gamma^t \vartheta^0 \qquad t = 0, 1, \ldots, t_1,$$

where $\gamma > 0$ is the rate of economic growth.

It should be stressed, that the fastest growth which can be achieved by the von Neumann economy equals its optimal technological effectiveness α_M. Thus, we may consider trajectory of intensity:

$$\overline{\vartheta}(t) = \alpha_M^t \overline{\vartheta} \qquad (9)$$

where $\overline{\vartheta}$ is a vector of intensity in the von Neumann optimal equilibrium (Panek 1997, 2000). The trajectory described by (9) is called the optimal steady growth intensity trajectory in the von Neumann model. The half line

$$N = \{\lambda\overline{\vartheta} \mid \lambda > 0\}$$

is called the turnpike or the von Neumann ray.

Field literature provides many theorems regarding so called, turnpike stability in the von Neumann-Leontief models with different input-output matrices *A* and *B* which are rectangular or square, semi-positive or containing negative elements (Czeremnych 1982, Intriligator 1971, Nikado 1968, Panek 1997, 2000). Whereas such models differ, sometimes significantly, the essence of the turnpike theorems proved on their basis, principly remains the same.

According to all such theorems, regardless of the initial state of the economy, the growth trajectories that are optimal in terms of a wide range of criteria, almost always, with the exception of a certain periods at the beginning and the end of a specified horizon T, run in any proximity to the turnpike. The longer the time horizon T, the more pronounced such a convergence.

Further, for the sake of simplicity, we will discuss a particular case of the von Neumann economy in which the following assumptions are additionally applied:

(3i) the input-output matrices A and B are $n \times n$ square matrices,

(4i) in the optimal state of von Neumann equilibrium $(\alpha_M, \overline{\vartheta}, \overline{p})$ the intensity and price vectors are positive and conditions (I) and (II) of Definition 3 are satisfied as equations:

$$\alpha_M \overline{\vartheta} A = \overline{\vartheta} B, \qquad \alpha_M A\overline{p} = B\overline{p}$$

(5i) the output matrix B is non-singular, matrices AB^{-1} and $B^{-1}A$ are non-negative, and all eigenvalues of matrix AB^{-1} are real and distinct.

Itssolution to the system of inequalities (6) with the initial condition (7) is ambiguous. In other words, under above assumptions, there are many feasible intensity trajectories and hence many feasible input and output trajectories, which satisfy (6) and (7). In order to point out a single specified trajectory, it is necessary to define a selection criterion. For instance, as growth criterion, we set the maximization of the value of production in the last period of horizon T, expressed in equilibrium prices. Hence we obtain the following problem:

$$\max \vartheta(t_1)B\overline{p}$$

subject to (6) and (7)

The solution $\{\vartheta^*(t)\}_{t=0}^{t_1}$ is called the optimal intensity trajectory. The trajectories $\{z^*(t)\}_{t=0}^{t_1}$ and $\{y^*(t)\}_{t=0}^{t_1}$ that correspond to it (in accordance with (8)) are called the optimal input and output trajectories.

Let the angular distance between a semi-positive intensity vector ϑ and the normalized optimal intensity vector $\overline{\vartheta} \in N$ be

$$d(\vartheta, N) = \left\| \frac{\vartheta}{\|\vartheta\|} - \overline{\vartheta} \right\|,$$

where $\|\vartheta\| = \sum_{i=1}^{n} |\vartheta_i|$ and $\|\overline{\vartheta}\| = 1$.

Definition 4. The von Neumann economy is globally stable if for any (arbitrarily small) number $\varepsilon > 0$ there is such a natural number l_ε (independent on t_1) that $t_1 \geq 2l_\varepsilon$ implies

$$d(\vartheta^*(t), N) < \varepsilon \quad \text{for} \quad l_\varepsilon \leq t \leq t_1 - l_\varepsilon .$$

Note here, that the longer is the time horizon T = {0, 1, …, t_1}, the longer is the interval for which $l_\varepsilon \leq t \leq t_1 - l_\varepsilon$ holds.

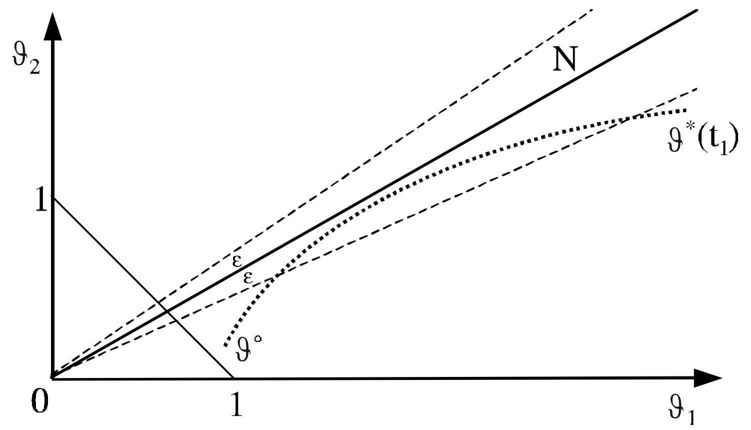

Fig. 2. Turnpike stability of the von Neumann economy in space R^2

It has been proved that under assumptions (i) - (5i), the von Neumann economy is globally stable (Czeremnych 1982, Panek 1997, 2000). The convergence of the optimal intensity trajectory $\{\vartheta^*(t)\}_{t=0}^{t_1}$ to the von Neumann ray N (the turnpike) is illustrated in Fig. 2.

Similar turnpike properties can be derived for optimal input and output trajectories. In other words, if we denote:

$$N^x = \{\lambda\bar{x} \mid \lambda > 0\}$$

$$N^y = \{\lambda\bar{y} \mid \lambda > 0\}$$

where, $\bar{x} = \bar{\vartheta}A$, $\bar{y} = \bar{\vartheta}B$, then for any number $\varepsilon > 0$ there is such a natural number l_ε (independent on t_1) that

$$d(z^*(t), N) < 0 \quad \text{and} \quad d(y^*(t), N) < \varepsilon \ \text{ for } l_\varepsilon \le t \le t_1 - l_\varepsilon .$$

4. Growth equilibrium in the Solow-Shell model. An application of the optimal control theory

We will proceed now from the maximally disaggregated competitive A-D-McK economy through the von Neumann partially disaggregated multi-product economy with a single (global) production space and one (global, production) growth criterion to the maximally aggregated single-product, double factor economy of the Solow-Shell type. The focus in this Section will be on resolving the issue of the so-called optimal distribution of income (between investment and consumption). Our growth criterion will be the maximization of per capita consumption utility in a specified time perspective.

As in Section 2, we will operate under assumption that time changes continuously and that the time horizon of the economy is a bounded interval $T = [0, t_1]$. We shall use the following notation (all variables and parameters are now scalars): $z(t)$ – employment at time t; $k(t)$, $y(t)$, $c(t)$ – the amounts of investment, income and consumption in period t; λ – the positive rate of population growth; μ – the positive rate of capital depreciation.

Assume that in horizon T:

– employment grows (autonomously) at the rate $\lambda > 0$:

$$z(t) = z^0 e^{\lambda t} \qquad (10)$$

– the capital growth is described by the differential equation

$$\frac{d}{dt} k(t) = i(t) - \mu k(t) \qquad (11)$$

with the initial condition

$$k(0) = k^0 > 0 \qquad (12)$$

– the size of income at time t is a function of capital stock and employment at that time; this relationship is described by the Cobb-Douglas production function homogenous of degree 1:

$$y(t) = a k^{\varepsilon}(t)\, z^{1-\varepsilon}(t) \qquad (13)$$

where $\varepsilon \in [0, 1]$ is the income elasticity with respect to capital and $1 - \varepsilon$ is the income elasticity with respect to labour,

– consumption $c(t) \geq 0$ is a fraction of income net of investment $i(t) \geq 0$:

$$c(t) = y(t) - i(t)\,. \qquad (14)$$

Let $s(t)$ be an investment rate defined as

$$s(t) = \frac{i(t)}{y(t)}\,.$$

The system (10)-(14) can be expressed then in the following equivalent form:

$$\frac{d}{dt}k(t) = as(t)k^{\varepsilon}(t)z^{1-\varepsilon}(t) - \mu k(t)$$

$$c(t) = a(1 - s(t))k^{\varepsilon}(t)z^{1-\varepsilon}(t) - \mu k(t) \qquad (15)$$

$$s(t) \in [0,1]$$

$$k(0) = k^0$$

The three functions *s*(*t*), *k*(*t*), *c*(*t*) that satisfy system (15) in T are said to describe the feasible growth process (in the Solow-Shell model). The functions *k*(*t*), *c*(*t*) are called the feasible trajectories of capital and consumption. The functions *y*(*t*), *i*(*t*) that correspond to the feasible process are called the feasible trajectories of income and investment. If the three functions *s*(*t*), *k*(*t*), *c*(*t*) satisfy system (15), then the four functions *i*(*t*), *k*(*t*), *y*(*t*), *c*(*t*) satisfy system (10)-(14) (and vice versa).

Let $U : R^1_+ \rightarrow R^1_+$ be a continuous and differentiable concave and increasing global (social) function of consumption utility.

The postulate of maximizing the utility of per capita consumption within time horizon T, can be expressed now as follows:

$$\max \int_T U\left(\frac{c(t)}{z(t)}\right)dt \qquad (16)$$

subject to (15)

The feasible process *s**(*t*), *k**(*t*), *c**(*t*) which constitutes a solution to this problem is the optimal growth process. Functions *k**(*t*), *c**(*t*) are called the optimal trajectories of capital and consumption. The corresponding functions *y**(*t*), *i**(*t*) are called the optimal trajectories of income and investment

The solution to problem (16) is obtained on the basis of the optimal control theory (Panek 1989). Its form depends on the initial level of capital stock k^0 and the length of horizon T. Further, as an example, we will present a solution to the problem (16) where the horizon is long and the initial capital stock is relatively low and satisfies the inequality:

$$k^0 < \bar{k} = z^0\left(\frac{a\varepsilon}{\mu}\right)^{\frac{1}{1-\varepsilon}}.$$

In this case, there are such a value of investment rate $\bar{s} \in (0,1)$ and moments of time $\tau_1, \tau_2 \quad (0 < \tau_1 < \tau_2 < t_1)$, that the solution to problem (16) is the following process:

$$s^*(t)=\begin{cases} 1 & \text{for} \quad t\in[0,\tau_1), \\ \bar{s} & \text{for} \quad t\in[\tau_1,\tau_2), \\ 0 & \text{for} \quad t\in[\tau_2,t_1], \end{cases}$$

$$k^*(t)=\begin{cases} \left\{\left[(k^0)^{1-\varepsilon}-d\right] e^{-\mu(1-\varepsilon)t}+de^{\lambda(1-\varepsilon)t}\right\}^{\frac{1}{1-\varepsilon}} & \text{for } t\in[0,\tau_1), \\ \bar{k}e^{\lambda t} & \text{for } t\in[\tau_1,\tau_2), \\ \bar{k}e^{(\mu+\lambda)\tau_2}e^{-\mu t} & \text{for } t\in[\tau_2,\tau_1], \end{cases}$$

$$c^*(t)=\begin{cases} 0 & \text{for} \quad t\in[0,\tau_1), \\ a(1-\bar{s})\bar{k}^{\varepsilon}(z^0)^{1-\varepsilon}e^{\lambda t} & \text{for} \quad t\in[\tau_1,\tau_2), \\ a\bar{k}^{\varepsilon}(z^0)^{1-\varepsilon}e^{[\lambda(1-\varepsilon)-\mu\varepsilon]t} & \text{for} \quad t\in[\tau_2,t_1], \end{cases}$$

where:

$$d=\frac{a(z^0)^{1-\varepsilon}}{\mu+\lambda}$$

The intervals $[0,\tau_1),[\tau_1,\tau_2),[\tau_2,\tau_1]$ are called initial, middle and final phases of growth. It is interesting that only the length of the middle phase $[\tau_1,\tau_2)$ grows with the lengthening of the time horizon T.

In the middle phase of growth the rate of investment $\bar{s}(t)=\bar{s}\in(0,1)$ and a steady growth of capital and consumption at the rate $\lambda > 0$ is exhibited:

$$\begin{aligned} \bar{k}(t)&=\bar{k}e^{\lambda t}, \\ \bar{c}(t)&=\bar{c}e^{\lambda t}, \end{aligned} \tag{17}$$

where $\bar{k}=z^0(a\varepsilon/\mu)^{1-\varepsilon}$, $\bar{c}=a(1-\bar{s})\bar{k}^{\varepsilon}(z^0)^{1-\varepsilon}$.

One can easily verify that in the middle phase also income and investment grow at the rate λ.

It must be stressed that, in this process of steady growth the maximum utility of consumption per capita is achieved at every moment of time *t*. Therefore we refer to it as the process of maximum steady growth in the Solow-Shell model.

The trajectories of capital, production, consumption and investment in this maximum process of steady growth play the role of (capital, production, consumption and investment) turnpikes, toward which optimal trajectories converge in the middle phase of appropriately long time horizon (Panek 1989). The economy reaches the turnpike at the end of the initial phase and leaves it at the end phase (Fig. 3).

Notice, that the Solow-Shell economy is globally stable in the sense of Definition 4, if the optimal trajectory of intensity is replaced with the optimal capital (production, consumption, investment) trajectory, while the von Neumann ray is replaced with a proper turnpike. In its middle phase, the optimal stationary process is characteristic of an economy in its highest growth equilibrium which may maintain for any length of time, thereby maintaining a steady rate of growth and ensuring the maximum utility of consumption at any moment of time.

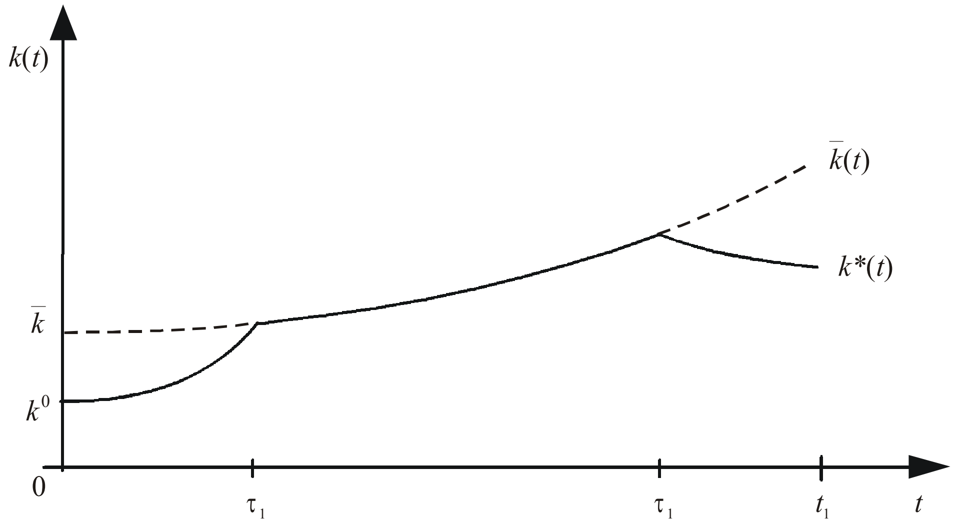

Fig. 3. An illustration of the convergence of optimal capital trajectory (solution to problem 16) with the capital artery

5. An attempt at a synthesis. The notion of dynamic σ- equilibrium

The modes of economies presented so far from the mathematical point of view are examples of stationary dynamic systems described with the use of differential equations of the type

$$\frac{d}{dt}x(t) = F\big(x(t)\big) \qquad (18)$$

(the A-D-McK model), difference inclusions of the type

$$\Delta x(t) \in \Phi\big(x(t)\big) \qquad (18a)$$

(the von Neumann model) and non-stationary dynamic systems described by differential equations of the type

$$\frac{d}{dt} x(t) = f\big(x(t), u(t), t\big) \qquad (18b)$$

(the Solow-Shell model), in which $x(t) = \big(x_1(t), ..., x_n(t)\big)$ is the so-called vector of the internal states of the system, whereas $u(t) = \big(u_1(t), ..., u_m(t)\big)$is the control vector at moment *t*.

The generally accepted definition of the state of equilibrium in such a system reads as follows (Panek 1989): an equilibrium state of the system is such an internal state $\bar{x}$ in which the system remains for any interval of time with zero control.

In the stationary systems (18) and (18a), this would be a vector $\bar{x}$ for which (respectively)

$$F(\bar{x}) = 0 \qquad (19)$$

or

$$0 \in \Phi(\bar{x}) . \qquad (20)$$

In such systems, the environment has no affect on their internal states, so by definition, their control is zero.

In a non-stationary system (18b), a state of equilibrium defined in such a way should be a solution $\bar{x}$ to the equation:

$$f(\bar{x}, 0, t) = 0 . \qquad (21)$$

Such a solution, however, (with the exception of trivial cases), does not exist. The fact of the matter is that the static equilibrium is typical for stationary systems.

In our case, it is only the first of the three models presented here, i.e. the A-D-McK model, for which the definition of the state of a (competitive) equilibrium is consistent with the definition of the state of a static equilibrium (19) adopted commonly in the theory of systems and in technical science. The Walrasian notion of competitive equilibrium, however, is based on a false hypothesis wherein there are economies that are: (a) stationary, (b) absolutely isolated, capable of functioning in complete isolation from the external world and requiring no external input of energy or information (zero control). Real-life economies are neither stationary nor absolutely isolated.

Can the theory of economic equilibrium be moved beyond the magic circle of stationary states? Before I attempt to answer this question, let me determine which similarily, common features are shared by all three of the equilibrium concepts. In all of them, equilibrium indicates the existence of certain economic constants. Whereas in the competitive equilibrium constants include the size of production put out by individual producers, the extent of consumer demand and prices. The constants in von Neumann equilibrium are the production growth rate, the production structure, etc. Similarily, technical objects in a state of equilibrium do not change some of their specific properties. The equilibrium of a pendulum, for instance, is its vertical position at zero velocity (the state of rest).

When discussing equilibria in growth models, we always refer to certain periods in which equilibria do or do not appear. Equilibria, therefore, are not associated with a single state of rest assumed by an economy but rather with a certain sequence of its states assumed over time, or a process of growth in which the economy is capable of reproducing certain qualities and properties. In the competitive equilibrium, we observe the reproduction of such characteristics as the size of production output, the extent of demand and the level of prices – the trajectories of these variables in the competitive equilibrium are constant; the constant in time in the von Neumann equilibrium is the rate of production growth and production structure (the production trajectory is an exponential function), etc. Being in equilibrium, physical objects also reproduce certain qualities. E.g. the vertical axis of a ship in the state of equilibrium always points to the center of the Earth while its angular velocity is zero. The coordinates of its trajectory may be e.g. the geographic location and the traveling speed.

The collection of qualities or attributes of a system's equilibrium may change in time, as a result of the operation of both external factors (adaptation) and the internal ones (self-adjustment). This is especially true for systems as complex as economies. A change in the type of the attributes of an equilibrium translates, in fact, into a change of equilibrium. In the case of complex systems, we should speak of different types of equilibria rather than of a single absolute equilibrium. In the light of the above remarks, we need a definition that would:

- generalize the classical definition of static equilibrium,
- bring relativity into the concept of equilibrium rather than associating it with a single "state of rest" of a system or ruling out the option of "external" impact to initiate a certain equilibrium,
- apply equally well to stationary and non-stationary systems.

In assuming that an equilibrium, however defined, should be characterized by the non-variability of certain attributes of the system, we do not impose the absolute condition whereby a part of such a characteristic is its internal state corresponding to zero control, i.e. a state of equilibrium as used in technical science. Generally speaking, such constants may include the values of certain functions of internal conditions and control describing specific qualities of a system. These need not be a system's internal states but rather, for example, their derivatives (in the case of a smooth system, the trajectory of equilibrium states is a linear function) or the sum of the values of their coordinates (if such an operation is acceptable), etc. In addition, it is desirable that a system in the state of equilibrium is characterized by a "regular" trajectory, which in the case of smooth systems described with the use of differential equations and inclusions, may translate into e.g. their continuity and differentiability. The above concepts have been incorporated into the following definition:

Definition 5. Let transfomation $\sigma : R^m \times R^n \to R^k$ be a measure, defined on internal states and controls, of those attributes of the system whose constant levels are thought of as constituting the symptoms of an equilibrium. We consider a system with the control trajectory $\bar{u}(t)$ and the state trajectory $\bar{x}(t)$ as remaining in equilibrium in horizon T if:

(i) $\sigma(u, x) \neq$ const. on $R^m \times R^n$ (the condition of non-triviality),

(ii) trajectories $\bar{u}(t), \bar{x}(t)$ are continuous and differentiable (the conditon of "regularity" of the system trajectories in a σ- equilibrium applies to smooth systems only),

(iii) $\sigma(\bar{u}(t), \bar{x}(t)) =$ const. at any moment (period) $t \in T$ (the condition of non-variability of σ- attributes of a system in a state of equilibrium).

In absolutely isolated systems described by systems of differential equations (18) and difference inclusions of type (18a), there is no control $\bar{u}(t)$. In other words, a system of type (18) (and respectively (18a)) with trajectory $\bar{x}(t)$ maintains σ- equilibrium in the sense of Definition 5, if:
(i) $\sigma(0, x) \neq$ const. on $R^m \times R^n$,
(ii) trajectory $\bar{x}(t)$ is continuous and differentiable (in the case of system (18)),
(iii) $\sigma(0, \bar{x}(t)) =$ const. for each $t \in T$

In practice, the form of transformation σ depends on the specific nature of the problem in question, e.g. the purpose of a study, etc. The classical definition of the state of a static equilibrium is obtained by assuming that $\bar{u}(t) = 0$ and $\sigma(u, x) = x$.

In system (4), the price trajectory $\bar{p}(t)$ in an A-D-McK economy maintaining a σ- equilibrium with the function

$$\sigma(u, p) = p$$

satisfies the condition $\bar{p}(t) = \bar{p}$ for $t \geq 0$.

The optimal stationary intensity trajectory $\{\bar{\vartheta}(t)\}_{t=0}^{t_1}$ in form (9) satisfying system (6a) describes the von Neumann economy in a σ- equilibrium with the function

$$\sigma(u, \vartheta) = \frac{\vartheta}{\|\vartheta\|},$$

where $\|\vartheta\| = \sum_{i=1}^{m} \vartheta_i$.

Similarily, the optimal stationary process of growth $\bar{s}(t), \bar{k}(t), \bar{c}(t)$ in form (17) describes the Solow-Shell economy maintaining a dynamic σ - equilibrium with the function

$$\sigma(s,k,c) = (s, \delta_k, \delta_c)$$

where:

$$\delta_{k(t)} = \frac{d}{dt} k(t) \cdot \frac{1}{k(t)}$$

$$\delta_{c(t)} = \frac{d}{dt} c(t) \cdot \frac{1}{c(t)}$$

The above definition of the σ - equilibrium applies both to absolutely isolated systems and to a relatively isolated system. In the light of this definition, there is no point in searching for an “absolute” economic equilibrium. The main advantage of the definition, however, is that it allows for extending the notion of equilibrium beyond the class of simple (technical) stationary systems and placing it in the realm of complex, non-stationary socio-economic systems.

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