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The non-stationary Arrow-Hurwich market

Abstract. In the paper, basing on the Arrow-Hurwich market, an attempt is made to prove that non-stationary economic system is not a real obstacle in investigating of its stability. It is shown that the classical equilibrium is not condition *sine qua non* for stability of the market.

Keywords: non-stationary market, equilibrium price, price trajectory, spherical stability, global stability.
JEL Codes: C62, D58.

1. Introduction

For a long time different attempts have been undertaken in mathematical economics to go beyond the static Walrasian equilibrium theory. In order to describe the real world phenomena more adequately, a long range of dynamic models have been elaborated and other alternative concepts of economic equilibria have been proposed (compare the neo-classical concept of balanced growth or the von Neumann dynamic equilibrium theory – works [1,2,4,7]).

Almost all theoretical dynamic models are stationary which, roughly speaking, implies that relations between their variables (their structure) remains constant over time. In such models equilibrium is still the primary notion, while stability is the secondary one.

The purpose of this paper is to show, based on the Arrow-Hurwich market, that non-stationarity of an economic system and hence the lack of its persistent equilibrium (no matter how understood) is not a real obstacle in defining and investigating stability. In other words, it is demonstrated that existence of equilibrium is not the *conditio sine qua non* for stability of the market.

2. The model of the market

We focus our attention on exchange processes which occur on a market at every moment t of an infinitely long time horizon $T = [0, +\infty)$. The object of exchange are n distinct consumer goods. The number of goods is constant in time. Goods are demanded or supplied by consumers. Contrary to the stationary Arrow-Hurwich model, the number of consumers, as well as the baskets of goods which they deliver to the market change over time.

At moment t, consumer $k\,(k=1,\ldots,m(t))$ buys a basket of goods $x^k(t) = \left(x_1^k(t), ..., x_n^k(t)\right)$. His outlays for that purpose cannot exceed his potential income equal to the value of basket $y^k(t) = \left(y_1^k(t), ..., y_n^k(t)\right)$ supplied by him at this moment. This upper limit of spending does not depend on whether he manages to sell the whole supplied basket or not, due to his buffer stock of money, which he can use to cover the gap between outlays and receipts from selling goods[1].

Prices of goods result from the relations between global supply and global demand and no singular consumer can influence them. As long as global demand for a certain good exceeds global supply – the price of this good increases. In the opposite case prices decrease.

Taking prices as given, consumers determine their demand in accordance with their preferences which can also vary in time. The preference relation of consumer k is described by his individual dynamic utility function $u^k(x, k)$, implying that the same bundle of goods x, considered at two different moments of time, may have different utility.

Transactions take place, generally out of any equilibrium, at every moment of time. If there is excess demand for certain goods some consumers are rationed and are not able to purchase the whole desired baskets of goods. This, however, does not induce them to buy substitutes. They would rather add the not spent part of income to their buffer stocks of money. Excess supply causes in turn involuntary stocks of unsold commodities which augment the content of baskets offered in subsequent periods.

Let $p(t)=(p_1(t), \ldots, p_n(t))$ be a price vector at moment t. Choosing the basket of goods at moment t, consumer k solves the following utility maximization problem[2]:

$$\max u^k(x, t)$$

subject to
$$\langle p(t), x\rangle \le I^k\left(p(t)\right), \tag{1}$$
$$x \ge 0,$$

[1] We assume that the consumers have such buffer stocks of money in order to present more comprehensive description of the functioning of the market. There is no need however to introduce them into the model explicitly.

[2] See [8].

where $I^k\left(p(t)\right)=\langle p(t), y^k(t)\rangle > 0$ is the potential income of consumer *k* from selling his basket of goods $y^k(t); \langle x, y\rangle = \sum_i x_i y_i$.

We assume that the dynamic utility function $u^k : R_+^{n+1} \to R^1 \left(k=1,\ldots,m(t)\right)$ satisfies assumptions (III)-(V) stated in [8]. Then the solution of problem (1) is a continuous on $\operatorname{int} R_+^{n+2}$ function φ^k of prices *p*, income and time *t*[3]:

$$x^k(t) = \arg\max_{\substack{\langle p(t),x\rangle \le I^k(p(t)) \\ x\ge 0}} u^k(x,t) = \varphi^k\left(p(t), I^k(p(t),t\right). \tag{2}$$

Since the potential income I^k is a function of prices itself, for simplicity we can write

$$\varphi^k\ \left(p(t), I^k(p(t),t\right) = f^k\ \left(p(t),t\right),\ \left(k=1,\ldots,m(t)\right).$$

Functions f^k have the same properties as functions φ^k.

Definition 1.
(i) The sum

$$f^d\left(p(t),t\right) = \sum_{k=1}^{m(t)} f^k\left(p(t),t\right)$$

is called a vector of total demand for goods at moment *t*, while the sum

$$f^s(t) = \sum_{k=1}^{m(t)} y^k(t)$$

is called a vector of total supply of goods at moment *t*.
(ii) Consequently, the difference

$$z\ \left(p(t),t\right) = f^d\ \left(p(t),t\right) - f^s(t)$$

is a vector of excess demand on the market at moment *t*.

Inequality $z_i\ \left(p(t),t\right) > 0$ implies that total demand for good *i* at moment *t* exceeds its total supply. Conversely, inequality $z_i\ \left(p(t),t\right) < 0$ reflects a surplus of supply over demand for good *i*.

[3] Ibid., theorem 10. Assumptions (III)-(IV) can be weakened e.g. the declining utility of a basket with respect to time is not necessary for the continuity of demand function.

[4] The details of this model are presented in [3].

Functions $f^d\ (p(t),t)$ and $z\ (p(t),t)$ are positively homogenous of degree zero with respect to prices, i.e.

$$\forall t \geq 0, \forall p > 0\ \forall \lambda > 0 \left(f^d(\lambda p,t) = f^d(p,t) \right), \left(z(\lambda p,t) = z(p,t) \right).$$

We further assume that the excess demand $z(p(t), t)$ function satisfies conditions:

(I) $\quad z \in C^1\left((R_+^n \setminus \{0\} \times R_+^1 \to R^n \right)$

(II) $\quad \forall t \geq 0, \forall i \left(p_i = 0 \Rightarrow z_i(p,t) > 0 \right)$.

The first condition of continuity and differentiability has a formal character. The second condition implies that global demand for any good offered for free always exceeds global supply.

Denote by $\sigma(t)$ a scalar function of time satisfying condition

(III) $\quad \sigma \in C^0\left(R_+^1 \to R_+^1 \right)$ and $\inf \sigma(t) = \bar{\sigma} > 0$.

Let

$$p(t) = \left(p_1(t), ..., p_n(t)\right),$$

$$\dot{p}(t) = \left(\dot{p}_1(t), ..., \dot{p}_n(t)\right),$$

$$z\ (p(t),t) = \left(z_1\ (p(t),t), ..., z_n\ (p(t),t)\right).$$

Definition 2. We call the system of differential equations

$$\dot{p}(t) = \sigma(t) z\ (p(t),t) \tag{3}$$

with a given initial condition

$$p(0) = p^0 > 0 \tag{4}$$

a non-stationary Arrow-Hurwich model.

Note, that for any moment t, the number $\sigma(t) > 0$ is a reaction coefficient of prices to disequilibrium between global supply and global demand.

Its worth stressing that in comparison to other standard systems of price dynamics, the distinctive feature of system (3) is that it describes adjustment of prices on time axis $T = [0, +\infty)$ in a non-stationary environment with excess demand function explicitly dependent on time (see e.g. [6], chap. 1).

Further, we assume that $\forall p^0 > 0$ system (3) with initial condition (4) has uniquely determined solution on the time axis $[0,+\infty)$. We denote it by p_T.

Definition 3. We call, defined on semi axis $T = [0, +\infty)$ positive solution to the system of differential equations (3) with initial condition (4) a (p^0, ∞) – a feasible trajectory of prices in the non-stationary Arrow-Hurwich market.

Generally, the non-stationary system of equations has no solution $\overline{p}_T$ satisfying condition $\overline{p}(t) = \overline{p} = \text{const}$, which is equivalent to

$$\forall t \geq 0 \left(z(\overline{p}, t) = 0 \right).$$

In other words, prices of equilibrium in their classical meaning as a fixed point of system (3) do not exist. Hence, the non-stationary Arrow-Hurwich market is never in a permanent equilibrium.

Theorem 1. Under assumptions (I)-(III), every solution p_T to the system of differential equations (3) with initial condition (1) has the following properties:
(1) $\forall t \geq 0$ $(p(t) > 0)$, i.e. it is positive on the whole semi axis $[0, +\infty)$,
(2) $\forall t \geq 0$ satisfies condition $\langle p(t), z\,(p(t), t)\rangle = 0$ (the Walras' Law)
(3) $\forall t \geq 0$ $\|p(t)\| = \|p^0\|$ i.e. it lies on n-dimensional sphere of radius $r = \|p^0\|$, centred at 0.

The proof of part (2) of the above theorem is an exact repetition of the proof of lemma 6.1, placed in [5], pp.108-109, while proofs of parts (1) and (3) can be found in [6] – lemmas 1.1, 1.2, pp.15-16.

3. Spherical stability of the non-stationary Arrow-Hurwich market

In the traditional analysis of a market we assume its stationarity. Investigating a stationary market, first of all we attempt to prove the existence of permanent equilibrium prices and reveal their properties. Only then we try to explore whether the market is stable, i.e. if it posses the ability to restore equilibrium.

We call stationary market globally asymptotically stable if any feasible trajectory of prices, starting from any initial price vector $p^0 > 0$, converges to the equilibrium price vector (defined up to the multiplicity of a scalar). In the case of local stability, this property of feasible trajectories of prices is bounded only to these trajectories which start from an appropriately small neighbourhood of equilibrium.

Obviously, both concepts of stability cannot be applied to the non-stationary markets because of their lack of permanent equilibrium. Therefore, we will introduce a new notion of stability which we call a spherical stability. For that purpose consider two feasible trajectories of prices p_T^1, p_T^2 which are the solutions to the differential equations (3) with initial conditions

$$p^1(0) = p^{01} > 0, p^2(0) = p^{02} > 0.$$

As a measure of distance ρ between price vectors $p^1(t), p^2(t)$ we take the Euclidian norm:

$$\rho\left(p^1(t), p^2(t)\right) = \left\|p^1(t) - p^2(t)\right\|.$$

Definition 4. We call the non-stationary Arrow-Hurwich market spherically asymptotically stable if it satisfies the following condition:

$$\forall p^{01} > 0, \forall p^{02} > 0 : \left\|p^{01}\right\| = \left\|p^{02}\right\| \left(\left\|p^1(t) - p^2(t)\right\| \to 0 \right).$$

To clarify the meaning of the above definition, remember that in accordance to Theorem 1, any feasible trajectory of prices lies on n-dimensional sphere centred at 0, whose radius is the initial price vector p^0. In view of this, the non-stationary Arrow-Hurwich market is spherically asymptotically stable if all (p^0, ∞) feasible trajectories of prices from the same sphere converge (though – not excluding each other – with different speed).

The next assumption is an adapted to the non-stationary system (3) standard assumption that all considered goods are "normal ", i.e. they are characterised by relatively high direct price elasticity of demand and low cross elasticity of demand (see e.g.. [5], pp.110-111, assumption (III)):

(IV) $\forall r > 0\ \forall k > 0\ \exists \varepsilon > 0\ \forall p^1 > 0\ \forall p^2 > 0 : \left\|p^1\right\| = \left\|p^2\right\| = r\ \&$

$$\&\ \left\|p^1 - p^2\right\| \geq k \quad \forall p \in [p^1, p^2] \quad \forall t \geq 0\left((p^2 - p^1)J(p,t)(p^2 - p^1)^T < -\varepsilon\right),$$

where $J(p,t) = \left(\dfrac{\partial z(p,t)}{\partial p}\right)_{(n,n)}$ is the Jacobi matrix of the excess demand function $z(p, t)$.

Lemma 1. Under assumptions (II), (IV)

$$\forall r>0, \forall k>0\, \exists \varepsilon>0 \;\; \forall p^1>0\; p^2>0: \left\|p^1\right\|=\left\|p^2\right\|=r\, \&$$

$$\&\; \left\|p^1-p^2\right\| \geq k \quad \forall t \geq 0 \left(\langle p^1, z(p^2,t)\rangle + \langle p^2, z(p^1,t)\rangle > \varepsilon\right).$$

Proof. Take $r>0$, $k>0$ and any two positive price vectors p^1, p^2 satisfying assumptions (II) and (IV). Let

$$p(\tau)=p^1+(p^2-p^1)\tau,\; \tau \in [0,1]$$

then $$p(0)=p^1,\, p(1)=p^2.$$

Consider a function $\phi \in C^1([0,1] \rightarrow R^1)$ $\phi(0)=0$ defined as follows

$$\phi(\tau)=\left\langle p^2-p^1, z\,(p(\tau),t)-z\,(p^1,t)\right\rangle.$$

By assumption (IV)

$$\exists \varepsilon>0 \quad \forall \tau \in [0,1]\left(\frac{d}{d\tau}\phi(\tau)=(p^2-p^1)J(p(\tau),t)(p^2-p^1)^T<-\varepsilon\right)$$

which implies

$$\phi(1)=\left\langle p^2-p^1, z\,(p^2,t)-z\,(p^1,t)\right\rangle<-\varepsilon.$$

We complete the proof noting that by Walras' Law

$$\left\langle p^1, z\,(p^1,t)\right\rangle=0 \text{ and } \left\langle p^2, z\,(p^2,t)\right\rangle=0.$$

Theorem 2. Under assumptions (I)-(IV) the Arrow-Hurwich market is spherically asymptotically stable.

Proof. Take such two feasible trajectories of prices p_T^1, p_T^2, satisfying differential equations (3), that

$$p^1(0)=p^{01}>0,\; p^2(0)=p^{02}>0, \left\|p^{01}\right\|=\left\|p^{02}\right\|=r.$$

Assume, that $p^{01} \neq p^{02}$, then

$$\forall t \geq 0 \left(p^1(t) \neq p^2(t) \right).$$

Define a function

$$V(t) = \frac{1}{2} \left\| p^1(t) - p^2(t) \right\|^2 .$$

By lemma 1, its derivative

$$\dot{V}(t) = \langle p^1(t) - p^2(t), p^1(t) - p^2(t) \rangle$$

satisfies on semi-axis $[0, +\infty)$ condition:

$$\dot{V}(t) = \sigma(t) \left(< p^1(t), z(p^2(t), t) > + < p^2(t), z(p^1(t), t) > \right) < 0 .$$

The function $V \in C^1 \left(R^1_+ \to R^1_+ \right)$ is decreasing and non-negative on its domain, so

$$\lim V(t) = \bar{V} \geq 0 .$$

Assume, that $\bar{V} > 0$. Then

$$\exists k > 0 \; \forall t \geq 0 \quad \left(\left\| p^1(t) - p^2(t) \right\| \geq k \right)$$

and according to the lemma, there is such $\varepsilon > 0$ that

$$\forall t \geq 0 \left(\dot{V}(t) < -\bar{\sigma}\varepsilon \right),$$

which implies

$$0 \leq V(t) < V(0) - \bar{\sigma}\varepsilon t \to -\infty \;\text{ as }\; t \to +\infty .$$

This, however, is not possible. The obtained contradiction completes the proof.

It is worth noticing that the investigation of spherical stability (in the sense of definition 3) of a non-stationary market resolves itself into investigation of its stability on a unit sphere. In fact, take any (p^0, ∞) – feasible trajectory of prices $p(t)$, satisfying system (3). Let $r = \left\| p^0 \right\| > 0$. Then

$$\tilde{p}(t) = \frac{p(t)}{\left\| p(t) \right\|} = \frac{1}{r} p(t) ,$$

where $\tilde{p}(t)$ is a normalized trajectory of prices, $\left\| \tilde{p}(t) \right\| = 1$.

Consequently

$$\dot{\tilde{p}}(t) = \frac{1}{r}\dot{p}(t) = \frac{1}{r}\sigma(t)z\ \left(p(t),t\right) = \tilde{\sigma}(t)z\ \left(\tilde{p}(t),t\right),$$

where $\tilde{\sigma}(t) = \frac{1}{r}\sigma(t)$.

The non-stationary market described by vector equation of price dynamics

$$\dot{\tilde{p}}(t) = \tilde{\sigma}(t)z\ \left(\tilde{p}(t),t\right) \tag{5}$$

with initial condition

$$\tilde{p}(0) = \tilde{p}^0 > 0, \qquad \left\|\tilde{p}^0\right\| = 1 \tag{6}$$

is spherically asymptotically stable on n-dimensional sphere of radius $r = 1$ under the same assumptions which guarantee spherical stability of the initial market, given by (3) and (4).

Note that the system (5) is an "equation of motion" of the projection of trajectory of prices from n-dimensional sphere of radius r on a unit sphere.

4. Global stability of the non-stationary market with relative prices

According to theorem 2 only trajectories from the same n-dimensional sphere are convergent to each over. This excludes convergence of trajectories with different length of vectors of initial prices (at t = 0), i.e. these trajectories, satisfying system (3), for which $p^1(0) = p^{01} > 0$, $p^2 > 0$ and $\left\|p^{01}\right\| \neq \left\|p^{02}\right\|$. Therefore, one cannot speak of global asymptotical stability of the non-stationary Arrow-Hurwich market with absolute prices. On the other hand, due to the homogeneity of degree zero of the excess demand function, we have the same vector of excess demand for any given price vector p and its multiplicity. In particular we may consider vector of relative prices

$$\tilde{p} = \frac{1}{p_n} \cdot p = \left(\frac{p_1}{p_n}, \frac{p_2}{p_n}, \ldots, 1\right) = \left(\hat{p}, 1\right)$$

where $\hat{p} = \left(\hat{p}_1, \ldots, \hat{p}_{n-1}\right)$. Note that the last, n-th good serves here as a *numéraire* which is used to express prices of the remaining goods.

Let

$$\hat{z}(\hat{p},1,t)=\left(z_1(\hat{p},1,t),...,z_{n-1}(\hat{p},1,t)\right)$$

be an excess demand function with relative prices.

Further, instead of the non-stationary system (3) with initial condition (4), we will consider the following system of n-1 differential equations

$$\dot{\hat{p}}(t)=\sigma(t)\hat{z}(\hat{p}(t),1,t) \tag{3'}$$

with an initial condition

$$\hat{p}(0)=\hat{p}^0>0 \tag{4'}$$

describing dynamics of relative prices in the non-stationary Arrow-Hurwich market.

Definition 5. We call, a defined on semi axis $T = [0, +\infty)$ positive solution to the system of differential equations (3') with initial condition (4') a $\left(\hat{p}^0,\infty\right)$ – a feasible trajectory of relative prices in the non-stationary Arrow-Hurwich market.

To explain the relation between trajectories of prices $p_T, \hat{p}_T$ – the solutions of systems (3) and (3') – consider a simple (stationary) model of the Arrow-Hurwich market with two goods and the following price dynamics[4]:

$$\begin{aligned}\dot{p}_1(t) &= A\frac{p_2(t)}{p_1(t)} - B\\ \dot{p}_2(t) &= B\frac{p_1(t)}{p_2(t)} - A\end{aligned} \tag{7}$$

with A, $B > 0$.

The model satisfies assumptions (I) – (IV). Also the Wa`lras Law is valid. Hence

$$\forall t\geq 0\left(p_1(t)\left(A\frac{p_1(t)}{p_2(t)}\right)+p_2\left(B\frac{p_1(t)}{p_2(t)}-A\right)=0\right).$$

Consider an initial price vector $p^0 > 0$. The feasible trajectory of prices which starts from p^0 remains on the circle of a radius $r=\left\|p^0\right\|$ whose length is the norm of the initial price vector. If $t \to +\infty$, then all trajectories from the same circle (starting from different initial points lying on that circle) are asymptotically convergent to the equilibrium price vector $\overline{p} = \lambda(A, B), \lambda > 0, \|A, B\| = \frac{r}{\lambda}$, so that they converge asymptotically to each other. This convergence is a particular example (on a circle) of spherical stability in the sense of definition 4 (see. Figure 1).

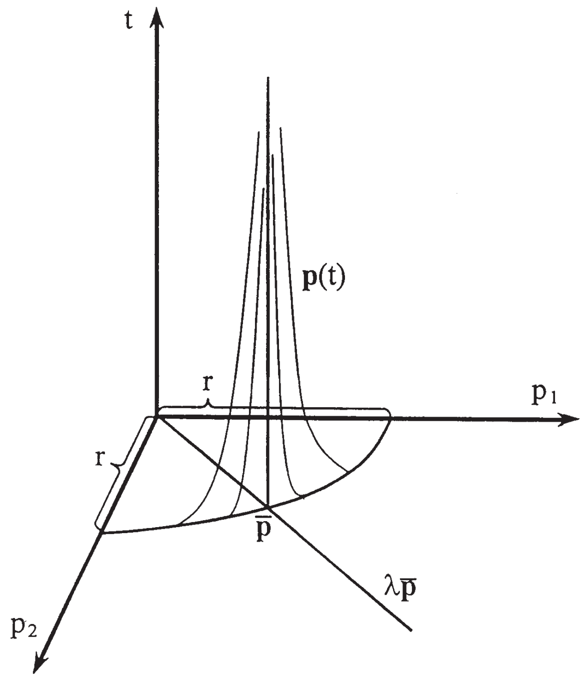

Figure 1. Spherical (on a circle) asymptotic convergence of absolute prices in the stationary Arrow-Hurwich market with two goods and equilibrium price vector $\overline{p}$ in the phase space R_T^3 (price space R_T^2 with added axis of time R_T^1)

Now, if we take the second good as a *numéraire*, the system of equations (7) resolves itself into a single equation:

$$\dot{\hat{p}}(t) = A\frac{1}{\hat{p}_1(t)} - B$$

which describes the dynamics of price of the first good in terms of units of the second good.

Every solution of this equation under any initial condition $\hat{p}_1(0) = \hat{p}_1^0 > 0$, for $t \to +\infty$ is asymptotically convergent to the point $\frac{A}{B}$. Obviously, this implies that any two positive solutions of this equation converge.

Moreover, since $\hat{p}_1(k) \to \frac{A}{B}$ we have

$$z_1(\hat{p}_1(t),1) = A\frac{1}{\hat{p}_1(t)} - B \underset{t}{\to} 0,$$

$$z_2(\hat{p}_1(t),1) = B\hat{p}_1(t) - A \underset{t}{\to} 0.$$

The above remarks, although they concern a stationary market, suggest the following definition:

Definition 6. We call the non-stationary Arrow-Hurwich market with relative prices globally asymptotically stable if any two feasible trajectories of its relative prices $\hat{p}_T^1, \hat{p}_T^2$ satisfy condition:

$$\left\| \hat{p}^1(t) - \hat{p}^2(t) \right\| \to 0 \quad \text{as} \quad t \to +\infty.$$

The transition from the non-stationary system of n equations of absolute prices dynamics (3) to the system of n-1 equations of relative prices dynamics (3') necessitates reformulation of assumption (IV) which states "weak" negative definiteness of the Jacobi matrix of the excess demand function in n-dimensional price space.

Its current version, adapted to system (3'), is the following:

(IV') $$\forall k > 0\, \exists \varepsilon > 0\, \forall \hat{p}^1 > 0\, \forall \hat{p}^2 > 0 : \left\| \hat{p}^1 - \hat{p}^2 \right\| \geq k$$

$$\forall \hat{p} \in [\hat{p}^1, \hat{p}^2], \forall t \geq 0 \left((\hat{p}^2 - \hat{p}^1) J(\hat{p},1,t)(\hat{p}^2 - \hat{p}^1)^T < -\varepsilon \right),$$

where

$$J(\hat{p},1,t) = \left(\frac{\partial \hat{z}(\hat{p},1,t)}{\partial \hat{p}} \right)_{(n-1,n-1)}.$$

Theorem 3. Under assumptions (I)-(III) and (IV') the non-stationary Arrow-Hurwich market with relative prices is globally, asymptotically stable.

Proof. The proof is similar to that of theorem 2. Taking any $k > 0$ and such positive vectors of prices $\hat{p}^1, \hat{p}^2$, that $\left\| \hat{p}^1 - \hat{p}^2 \right\| \geq k$, we repeat (under assumption (IV')) the proof of lemma 1 and come to the conclusion, that

$$\exists \varepsilon > 0 \quad \forall t \geq 0 \left(\langle \hat{p}^1, \hat{z}(\hat{p}^2,1,t) \rangle + \langle \hat{p}^2, \hat{z}(\hat{p}^1,1,t) \rangle > \varepsilon \right).$$

Take next any two feasible trajectories of prices $\hat{p}_T^1, \hat{p}_T^2$ – the solutions of system (3') – with initial conditions

$$\hat{p}^1(0) = \hat{p}^1 > 0, \hat{p}^2(0) = \hat{p}^{02} > 0, \hat{p}^{01} \neq \hat{p}^{02}$$

(if $\hat{p}^{01} = \hat{p}^{02}$, then $\hat{p}^1(t) = \hat{p}^2(t)$ for every $t \geq 0$).

Define a function

$$V(t) = \frac{1}{2}\left\|\hat{p}^1(t) - \hat{p}^2(t)\right\|^2.$$

Obviously, $V(t) \geq 0$ for $t \geq 0$. Simultaneously

$$\dot{V}(t) = -\sigma(t)\left(\langle \hat{p}^1, \hat{z}(\hat{p}^2(t),1,t)\rangle + \langle \hat{p}^2(t), \hat{z}(\hat{p}^1(t),1,t)\rangle < 0\right).$$

Function V is continuous (and differentiable), non-negative and decreasing on semi-axis $[0,+\infty)$, hence

$$\exists \bar{V} \geq 0 \left(\lim_t V(t) = \bar{V}\right).$$

Assume that $\bar{V} > 0$, then

$$\exists k > 0 \quad \forall t \geq 0 \left(\left\|\hat{p}^1(t) - \hat{p}^2(t)\right\| \geq k\right)$$

which in view of assumptions (I) and (IV') implies that

$$\exists \varepsilon > 0 \quad \forall t \geq 0 \quad \left(\dot{V}(t) \leq -\bar{\sigma}\varepsilon\right).$$

Hence $V(t) \to -\infty$, as $t \to +\infty$, as, which is impossible. The above contradiction leads the proof to its conclusion.

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