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Piotr MAĆKOWIAK
The Poznań University of Economics

Sensitivity of optimal paths with respect to horizon length in a nonstationary multisector growth model

Abstract: In the paper we present some results on convergence properties of optimal programs of growth in a multisector growth model context. There is no need for convexity or monotonicity of production mappings though we impose continuity in product topology on intertemporal social utility function. We show that if final stock is sustainable then sensitivity and continuity of optimal growth paths hold - both in investment and consumption. We apply our results to a simple multisector Leontief growth model with consumption and discounted utility.

Keywords: nonstationary model, optimal growth, sesitivity, continuity.
JEL codes: C61.

1. Introduction

The paper investigates two questions stated in optimal growth models: How close to each other are optimal programs if planning horizon length changes? Do the finite horizon optimal programs approximate some infinite horizon optimal program "reasonably" well? If both of these questions have positive answers then it is true that if social planner has a properly long but not infinite planning horizon length then the optimal programs do not cause much loss of social utility in the initial periods, in comparison to optimal programs with a longer planning horizon. In our study we use a model in which we do not require any kind of convexity – this is not very common in the literature – and using a mixture of methods from papers Dutta (1993), Nermuth (1979) we answer positively both of the above questions. Section two provides notation. In section three we introduce basic assumptions and definitions and discuss them. In section four we prove some needed lemmata. Section five presents the main results of the paper. In section six we apply our results to a simple linear growth model.

2. Notation

Let $S = \mathrm{R}_+^l$ denote the nonnegative orthant of l-dimensional real vector space R^l equipped with euclidean norm and N = {1, 2,...}. For all $t \in N$ $S_t := S$ and $S_c^T := \prod_{t=1}^{T} S_t$, $S_x^T := S_c^T$, $S_c := S_c^\infty = \prod_{t=1}^{\infty} S_t$, $S_x := S_c$. If $c \in S_c^T$ $\left(c \in S_c\right)$ it means $c = \{c_t\}_{t=0}^{T}$ $\left(c = \{c_t\}_{t=0}^{\infty}\right)$. Analogously for members of S_x and S_c^T. For $x, y \in S$ $x \geq y$ means $x_i \geq y_i$, $i = 1,\ldots, l$; $x \gneq y$ means $x \geq y$ and $x \neq y$.

3. The model

We present a version of the model described in Dutta (1993) and Nermuth (1979). *Ft* represents production mapping whose argument is an input and value is an output available from the given input after a fixed unit of time. We take the following assumptions on $F_t : \forall t \in \mathrm{N}$

(1) $F_t : S \to 2^S$ is upper hemicontinuous and compact valued;

(2) $\forall x \in S$ $F_t(x) \neq \varnothing$, $F_t(0) = \{0\}$;

(3) $\forall \varepsilon > 0 \exists \delta > 0 : 0 < \|x\| < \delta \Rightarrow \exists y \in F_t(x) : 0 < \|y\| < \varepsilon$;

(4) $x \gneq 0$ and $0 \in F_t(x) \neq \{0\} \Rightarrow \forall \varepsilon > 0$ $\exists y \in F_t(x) : 0 < \|y\| < \varepsilon$.

The last two assumptions are independent.

Example 1. Let $F : \mathrm{R}_+ \to 2^{\mathrm{R}+}$

$$\forall x \in \mathrm{R}_+ F(x) := \{0\} \cup \{x\}.$$

F fulfills conditions 1, 2 *and* 3 *but not* 4.

Example 2. Let $F : \mathrm{R}_+ \to 2^{\mathrm{R}+}$

$$\forall x \in \mathrm{R}_+ F(x) := \begin{cases} \{0\} & x \in [0,1] \\ [0, x-1] & x > 1 \end{cases}$$

F fulfills conditions 1, 2 *and* 3 *but not* 4.

The class of multifunctions fulfilling 1–4 is not empty.

Example 3. Let *A* be a positive square matrix of size *n*. Let $F : \mathrm{R}_+^n \to 2^{\mathrm{R}_+^n}$

$$\forall x \in \mathrm{R}_+^n F(x) := \{y \in \mathrm{R}_+^n : Ay \leq x\}.$$

We do not assume any convexity of images of F_t – the sets $F_t(S)$ are even not necessarily connected. In papers by Dutta (1993), Nermuth (1979) assumptions on F_t guarantee connectedness of images.

The first two assumptions are standard in multisector growth literature. Assumptions (3), (4) are fulfilled for example in the case when technology is productive (i.e. from a nonzero input we can get a nonzero output) and a possibility of costless waste is assumed. It should be noted that all the above assumptions are implied by standard assumptions in multisector growth models (see McKenzie (1986)). In the paper by Nermuth (1979) assumption (2) is not taken whereas the other assumptions are in at least a stronger form. Although the main proposals in papers of Dutta (1993), Nermuth (1979) refer to Brock's article Brock (1971), they do not use assumption (2).

Fix some $x_0 \in S$. *Feasible process* starting from a given x_0 is a sequence $(c,x) \in S_c x S_x$ such that $\forall t \in \mathrm{N}\; c_t + x_t \in F_{t-1}\left(x_{t-1}\right)$ where c_t, x_t denote consumption and capital stock inputs in period t respectively. Denote the set of all feasible processes by P (in what follows we keep x_0 fixed). Futher we assume that $S_c \times S_x$ and all its subsets (to be defined) are equipped with product topology (pointwise convergence topology). We call a vector $b \in S$ *attainable* if there exists $T \in N$ such that for some feasible process $(x,c) \in P$ holds $x_T \geq b$. We call a vector $b \in S$ *strongly attainable* (Nermuth (1993)) if it is attainable and $\forall 0 \neq x \in S \forall T \in \mathrm{N}\; \exists \tau \in \{0,1,2,\ldots\}$: $\exists \{\mathrm{x_t}\}_{t=T}^{T+\tau}\, x_T = x, x_{t+1} \in F_t(x_t), x_{T+r} \geq b$.We call a vector $b \in S$ *sustainable* if it is strongly attainable and there exists $T \in \mathrm{N} \forall \mathrm{t} \geq T\; \forall x \geq b : b \in F_t(x)$. We denote the set of all sustainable vectors by P_S. It follows that $0 \in P_S$. Define

$$\forall b \in Ps \forall t \in \mathrm{N} \quad P_b(t) := \{(c,x) \in P : x_t \geq b\}, \qquad P_b(\infty) := P,$$

$$P'_b(T) = \mathrm{proj}_{S_x^T \times S_x^T} P_b(T) \times \left((0,0),(0,0),...\right), \qquad P'_b(\infty) := P.$$

A pair (x, c) belongs to $P'_b(T)$ if there exists a pair $(c',x') \in P_b(T)$ such that $c_r = c'_r, x_r = x'_r, r = 1,...,T$ and $c_r = x_r = 0, r \geq T+1$. We can treat P_b, P'_b as mappings from $\mathrm{N} \cup \{\infty\}$ into $2^{S_c \times S_x}$. To consider their continuity we need to introduce a topology on their domains. On the set $\mathrm{N} \cup \{\infty\}$ define a base for a topology of sets of form (Nermuth (1979))

$$\{t\},\, t = 0,1,\ldots,$$

$$Z_T := \{t \in \mathrm{N}\!: \mathrm{t} \geq T \text{ or } t = \infty\}, T \in \mathrm{N}.$$

It is easy to see that the base defines a Hausdorff topology. We also introduce an intertemporal social utility function

$$V : S_c \to \mathrm{R},$$

which is continuous with respect to product topology on S_c. The arguments of V are consumption streams $c = (c_1, c_2, \ldots)$ where each c_t represents consumption in period t. It is not very common to use such a form of intertemporal utility function in multisector models without any assumption of time separability but our considerations do apply for time additive separable utility models which are standard in the optimal growth theory. We stress that domain of $V(\cdot)$ is S_c since Dutta's example 3.1 in Dutta (1993) invalidates our main results on sensitivity and continuity of optimal paths if $V(\cdot)$ is not continuous on S_c.

4. Continuity lemmata

Lemma 1. $\forall \in > 0 \forall t, T \in \mathrm{N}\, \exists \{x_r\}_{\tau=t}^{t+T} \in S_x^{T+1} : \quad \forall \tau = t, \ldots, T \quad x_{\tau+1} \in F_\tau(x_r),$ $\forall \tau = t, \ldots, T+1 \quad 0 < \|x_\tau\| < \varepsilon.$

Proof. The lemma will be proven by induction with respect to T. Fix any $t \in \mathrm{N}$ and $T = 1$. By assumption 3 for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x_t \in S,\ 0 < \|x_t\| < \delta$ there exists $x_{t+1} \in S, 0 < \|x_{t+1}\| < \varepsilon : x_{t+1} \in F_t(x_t)$. W.l.o.g. we can take $x_t \in S, 0 < \|x_t\| < \varepsilon,$ such that some of it coordinates equals zero. Assume that the thesis is true for T. We will show that it holds for $T + 1$. Suppose that $\{x_\tau\}_{\tau=t}^{t+T}$ fulfills $x_{\tau+1} \in F_\tau(x_\tau), 0 < \|x_\tau\| > \varepsilon_1$ for some $\varepsilon_1 > 0$. Fix any $\varepsilon > 0$. By taking $\varepsilon_1 > \varepsilon$ we find that there exists (by assumption 3) $x_{T+1} \in F_{t+T}(x_T) : 0 < \|x_T\| < \varepsilon$ which proves the lemma.

We fix $x_0 \in S$. For a proof of the next lemma see Nermuth (1979).

Lemma 2. Sets $P_b(T)$ are nonempty and compact in product topology for all $T \in \mathrm{N} \cup \{\infty\}, b \in P_s$.

Lemma 3. For any fixed $b \in P_s$, $P_b(\cdot)$ is a continuous mapping.

Proof. If $b = 0$ then $\forall T \in \mathrm{N} \cup \{\infty\}\colon\ \mathrm{P_b}(T) = P$, and $P_b(\cdot)$ is continuous. Assume that $b \neq 0$. To prove that $P_b(\cdot)$ is continuous at any $t \in \mathrm{N} \cup \{\infty\}$ it is sufficient and necessary to show that it is upper hemicontinuous and lower hemicontinuous at any t. Since $\mathrm{N} \cup \{\infty\}$ and P are first countable and Hausdorff it is equivalent to show that any $\{t_n\}_{n=1}^{\infty}, t_n \xrightarrow{n} t$ and $\{(c^n, x^n)\}_{n=1}^{\infty}, (c^n, x^n) \in P_b(t_n)$ there exists a convergent subsequence (we do not change indices) $(c^n, x^n) \xrightarrow{n} (c, x) \in P_b(t)$ and for any $(c, x) \in P_b(t)$ and any $\{t_n\}_{n=1}^{\infty}$, $t_n \xrightarrow{n} t$ there exist $(c^n, x^n) \in P_b(t_n)$, $n \in \mathrm{N}$ such that $(c^n, x^n) \xrightarrow{n} (c, x)$ in product topology (see Aliprantis and Border (1999, p. 534, theorem 16.21)). From the definition of the topology on $\mathrm{N} \cup \{\infty\}$ it is easily seen that a possible lack of continuity occurs only at $t = \infty$. Suppose that $\{t_n\}_{n=1}^{\infty}$, $t_n \xrightarrow{n} \infty$ and $\{(c^n, x^n)\}_{n=1}^{\infty} \in P_b(t_n)$. Since $\forall n\ P_b(t_n) \subset P_b(\infty) = P$ and P is compact then using Cantor diagonal process we can choose a convergent subsequence. Obviously, the limit process belongs to $P_b(\infty) = P$ – the upper hemicontinuity follows. Assume now that $(c, x) \in P$, where $c = \{c_t\}_{t=1}^{\infty} \in S_c$, $x = \{x_t\}_{t=1}^{\infty} \in S_c$, $\{t_n\}_{n=1}^{\infty}$, $t_n \xrightarrow{n} \infty$ and $t_{n+1} > t_n$. If $\forall t \in \mathrm{N}\ \mathrm{c_t} + x_t \neq 0$ then under assumption (2) $\forall t \in \mathrm{N}\ x_t \gneq 0$. By sustainability of b for every n = 1, 2,… there exists the smallest nonnegative integer T_n and a sequence $\{x'^n_\tau\}_{\tau=t_n}^{t_n+T_n} \in S_x^{T_n+1}$ such that $\forall \tau = t_n, ..., t_n + T_n : x'^n_\tau = x_\tau, x'^n_{t_n+T_n} \geq b$ and $x'^n_\tau \in F_{\tau-1}(x'^n_{\tau-1})$. It follows that $\forall n$ a feasible process $(\bar{c}^n, \bar{x}^n)$ defined by

$$(\bar{c}^n_\tau, \bar{x}^n_\tau) := \begin{cases} (c_\tau, x_\tau) & r = 1, ..., t_n \\ (0, x'^n_\tau) & r = t_n + 1, ..., t_n + T_n - 1 \\ (0, b) & r = t_n + T_n, t_n + 1, ... \end{cases}$$

is feasible though it is possibly not in $P_b(t_n)$. Let $n_0 = 1$ and define recursively n_j, j = 1, 2, 3,… as follows: n_{j+1} is the greatest nonnegative integer for which $n_{j-1} + 1 \leq n \leq n_j$ implies $(\bar{c}^{n_{j-1}}, \bar{x}^{n_{j-1}}) \notin P_b(t_n)$. We have that sequence $(c^n, x^n) := (\bar{c}^{n_{j-1}}, \bar{x}^{n_{j-1}})$, $n_{j-1} + 1 \leq n \leq n_j, j = 1, 2, \ldots$ fulfills $\forall n (c^n, x^n) \in P_b(t_n)$ and $(c^n, x^n) \xrightarrow{n}$ (c, x).

Suppose now that for the previously fixed $(c, x) \in P$ there exists $t \in \mathrm{N}$ such that $c_t + x_t = 0$. It implies that $\forall t' \geq t : c_{t'} = x_{t'} = 0$. Let T_0 denote the first period such that $x_{T_0} \neq 0$ and $\forall t \geq T_0 + 1 : x_t = 0$. W.l.o.g. we may assume that $\forall n : t_n > T_0$. From lemma 1, sustainability of b for any $\varepsilon > 0$ and for each $n \in \mathrm{N}$ we can construct a sequence $\{x'^n_T\}_{T=T_0+1}^{t_n+T_n}$ such that $\forall \tau \in \{T_0 + 1, \ldots, t_n\} : 0 < \|x'^n_\tau\| < \varepsilon t_n^{-1}$ and $\forall \tau \in \{T_0 + 1, \ldots, t_n + T_N\}\ x'^n_{\tau+1} \in F_\tau(x'^n_\tau), x'^n_{t_n+T_n} \geq b$ where for each n T_n is the smallest nonnegative number for which it is possible to reach b starting from $x'^n_{t_n}$. Now fix ε so small that $x'^n_{T_0+1} \in F_{T_0}(x_{T_0}), \|x'^n_{T_0+1}\| < \varepsilon t_n^{-1}$ (this is possible by assumption 4 and sustainability of $b \neq 0$) and $c_{T_0+1} \gneq x'^n_{T_0+1} \gneq 0$ if $c_{T_0+1} \neq 0$ (this can be done by the proof of lemma 1). Now define $\forall n \in \mathrm{N}$ a feasible process $(\bar{c}^n, \bar{x}^n)$ defined by

$$\left(\bar{c}_\tau^n, \bar{x}_\tau^n\right) := \begin{cases} \left(c_\tau, x_\tau\right) & \tau = 1, \ldots, T_0 \\ \left(c'_\tau, x''^n_\tau\right) & \tau = T_0 + 1 \\ \left(0, x''^n_\tau\right) & \tau = T_0 + 2, \ldots t_n + T_n, \\ \left(0, b\right) & \tau = t_n + T_n + 1, \ldots, \end{cases}$$

where $c'_{T_0+1} = 0$, if $c_{T_0+1} = 0$ and $c'_{T_0+1} = c_{T_0+1} - x''^n_{T_0+1}$ if $c_{T_0+1} \gneq 0$. Proceeding as previously we can find numbers n_j, $n_{j-1} < n_j$ and a sequence $\left(c^n, x^n\right) \xrightarrow{n} \left(c, x\right)$ such that $\forall n \left(c^n, x^n\right) \in P_b\left(t_n\right)$.

Lemma 4. $P'_b(\cdot)$ is a continuous mapping.

Proof. As in the previous lemma we have to show continuity only at $T = \infty$. Suppose that $t_n \xrightarrow{n} \infty, t_n < t_{n+1}$ and $\left(c''^n, x''^n\right) \in P'_b\left(t_n\right)$. By definition of $\left(c''^n, x''^n\right) \forall n \exists$ $\left(c^n, x^n\right) \in P_b\left(t_n\right) : \left(c''^n_\tau, x''^n_\tau\right) = \left(c^n_\tau, x^n_\tau\right), \tau = 1, \ldots T_n$. Since $P_b(\cdot)$ is u.h.c. we can choose a convergent subsequence (no change of indices) $\left(c^n, x^n\right) \xrightarrow{n} (c, x) \in$ $P_b\left(\infty\right)$. This is convergence in product topology (coordinatewise) so it follows that $\left(c''^n, x''^n\right) \xrightarrow{n} (c, x)$ and since $(c, x) \in P_b\left(\infty\right)$ it follows that $(c, x) \in P'_b\left(\infty\right)$ and is u.h.c. Suppose now that $\left(c', x'\right) \in P'_b\left(\infty\right) = P_b\left(\infty\right) = P$ and $t_n \xrightarrow{n} \infty$, $t_n < t_{n+1}$. From lower hemicontinuity of $P_b(\cdot)$ for each $n \exists \left(c^n, x^n\right) \in P_b\left(t_n\right)$ and $\left(c^n, x^n\right) \xrightarrow{n} \left(c', x'\right)$. Define now $\forall n \in \mathrm{N}$

$$c''^n := \left(c_1^n, c_2^n, \ldots, c_{t_n}^n, 0, 0, \ldots\right),$$

$$x''^n := \left(x_1^n, x_2^n, \ldots, x_{t_n}^n, 0, 0, \ldots\right).$$

By the above construction we see that $\forall n \left(c''^n, x''^n\right) \in P'_b\left(t_n\right)$ and $\left(c''^n, x''^n\right) \xrightarrow{n}$ $\left(c', x'\right)$ – lower hemicontinuity is proven.

5. Continuity and sensitivity of optimal paths

For each $T \in \mathrm{N} \cup \{\infty\}$ consider an optimization problem

$$\sup V\left(c'\right)$$

$$\left(c',x'\right)\in P'_b\left(T\right).$$

Whenever $T \in \mathrm{N}$ we have to do with finding the optimal consumption and utility level in a finite horizon. If $T = \infty$ the above problem is to find optimal consumption and utility level under infinite horizon. Let $\forall T \in \mathrm{N} \cup \{\infty\}$

$$\bar{V}\left(T\right) := \sup_{\left(c',x'\right)\in P'_b\left(T\right)} V\left(c'\right),$$

$$\bar{P}\left(T\right) := \left\{\left(c',x'\right)\in P'_b\left(T\right) : V\left(c'\right) = V\left(T\right)\right\}.$$

Theorem 1. $\bar{V} : \mathrm{N} \cup \{\infty\} \to \mathrm{R}, \bar{P} : \mathrm{N} \cup \{\infty\} \to 2^{S_c \times S_x}$ are well-defined continuous function and a u.h.c. mapping respectively.

Proof. From continuity of $V(\cdot)$ and lemma 3 it follows that the family of problems (1) is well defined i.e. the suprema are achieved. The second part of thesis is implied by Berge's Maximum Theorem (see Berge (1963, p. 116)).

Since $P \subset S_c \times S_x$ is equipped with the product topology, the product topology a base of neighborhoods of every point is given by a family of sets $\left(c,x\right)\in P$

$$U_{N,\varepsilon}\left(c,x\right) := \left\{\left(c',x'\right)\in P : \right.$$

$$\left.\left\|c_\tau - c'_\tau\right\| < \varepsilon, \left\|x_\tau - x'_\tau\right\| < \varepsilon, \tau = 1,\ldots,N\right\} N \in \mathrm{N}, \varepsilon > 0.$$

For a given $A \subset P, N \in \mathrm{N}, \varepsilon > 0$ we define an open neighborhood of A

$$U_{N,\varepsilon}\left(A\right) := \bigcup_{\left(c,x\right)\in A} U_{N,\varepsilon}\left(c,x\right).$$

Upper hemicontinuity of $\bar{P}$ means that for each open set $V \subset P, \bar{P}\left(\bar{T}\right) \subset V$ some $\bar{T} \in \mathrm{N} \cup \{\infty\}$ there exists a neighborhood Z of $\bar{T}$ such that $\bar{P}\left(T\right) \subset V, \forall T \in Z$. If we fix $N \in \mathrm{N}, \varepsilon > 0$ and define $V = U_{N,\varepsilon}\left(\bar{P}\left(\infty\right)\right)$ then V is an open neighborhood of $\bar{P}\left(\infty\right)$ so that there is a neighborhood Z_N of ∞ such that $\forall T \in Z_T : \bar{P}\left(T\right) \subset V$. W.l.o.g. we can assume that $Z_N = \left\{T_N\left(\varepsilon\right), T_N\left(\varepsilon\right)+1,\ldots\right\} \cup \{\infty\}$. We get

Corollary 1 (Turnpike Property). For any $b \in P_S$, every $N \in \mathrm{N}$ and every $\varepsilon > 0$ there exists a $T_N\left(\varepsilon\right) \in \mathrm{N}$ such that for every program $\left(c',x'\right)\in \bar{P}\left(T\right), T \geq T_N\left(\varepsilon\right)$, there is a $\left(c,x\right)\in \bar{P}\left(\infty\right)$ such that $\left\|c_\tau - c'_\tau\right\| < \varepsilon, \left\|x_\tau - x'_\tau\right\| < \varepsilon, \tau = 1,\ldots N$.

By the above corollary for every nonnegative integer N for every program (c, x) which brings maximal utility in a sufficiently large horizon T there is an infinte horizon optimal program whose first N-period truncation lies as close of optimal

T-period program as one wishes – this solves the continuity question (see Dutta (1993)).

Corollary 2. If $T_n \xrightarrow{n} \infty$ then $\bar{V}(T_n) \xrightarrow{n} \bar{V}(\infty)$.
From the corollary it follows that, independently of finite stock *b,* the optimal utility values are close to each other when the planning horizon is sufficiently long. One could show in addition that the convergence is monotonic i.e. $\bar{V}(T) \le \bar{V}(T+1)$ – this is a simple consequence of the definition of sustainable vectors and the form of optimization problems (1). However we should be cautious since the finite horizon problem 2 differs slightly from its analogue in the literature.

Example 4. Let $V : S_c \rightarrow \mathrm{R}$ and $V(c) = V_1(c_1) + V_2(c_2, c_3,...)$ where $V_i(\ldots)$, $i = 1, 2$ are continuous functions. In case of finite optimization for $T = 1$ we have

$$\sup\{V_1(c_1) + V_2(0,0,\ldots)\}$$

$$(c,x) \in P'_b(1),$$

while its literature counterpart is

$$\sup V(c_1)$$

$$(c,x) \in P'_b(1).$$

It is easily seen that maximands $\bar{c} = (\bar{c}_1, 0, \ldots)$ of the above problems are the same but it may be the case that $V_1(\bar{c}_1) > V_1(\bar{c}_1) + V_2(0,0,\ldots) = V(\bar{c}_1, 0, 0, \ldots)$ and even $V_1(\bar{c}_1) > V(c)$ for all $(c,x) \in P'_b(T), T \in \{2,3,\ldots\} \cup \{\infty\}$. In case of time additive separable utility function taking only nonegative values (periodwise) this 'anomaly' vanishes.

Nevertheless the above example does not invalidate any of our results.

Corollary 3. Assume uniqueness of the infinite horizon optimal program and fix some $N \in \mathrm{N}$. For sufficiently large horizons length $T \in \mathrm{N}$ optimal T-period programs are as close to each other in first N periods as one wishes.

Corollary 3 solves the sensitivity question (see Dutta (1993)).

6. An application

In this section we will apply our results to a simple dynamic Leontief model. Let A be a nonnegative, indecomposable and productive square matrix of size l[1]. Define $\forall x \in S$

$$F(x) := \{ y \in S : Ay \leq x \}$$

and let $\forall t = 0,1,\dots \; F_t = F$. It is easily seen that assumptions 1-4 are fulfilled by indecomposability and productivity of matrix A. Assume that there is one period utility function $U : S \to \mathrm{R}$ which is continuous, strictly concave and bounded. Define a multiperiod social utility function $V : S_c \to \mathrm{R}$:

$$\forall c \in S_c : V(c) = \sum_{t=1}^{\infty} \beta^{-t} U(c_t),$$

where $\beta > 1$ is a fixed discount factor. It is easy to see that $V(\cdot)$ is continuous in product topology. Fix an initial vector $x_0 \in S$. *A feasible path* is e sequence $\{c_t x_t\}_{t=1}^{\infty} \in S_c \times S_x$ such that $\forall t \; c_t + x_t \in F_{t-1}(x_{t-1})$. A feasible process (c, x) is called *optimal* if $V(c) \geq V(c')$ for any feasible process (c', x'). We see that 0 is a sustainable vector in this model. From decomposability and productivity of A it follows that there is a unique optimal process if it exists. Now we can use corollaries 1-3 to get desired conclusions on sensitivity and continuity characteristics of long-term planning. Results of Dutta (1993) and Nermuth (1979) do not apply here since their assumptions on production mappings are not fulfilled[2].

7. Conclusions

In the paper we proved under quite general assumptions that for a sufficiently long planning horizon the initial periods of optimal programs do not diverge from each other if there is a unique infinite horizon optimal program - this is valid for consumption and investment programs. We also showed that optimal utilities converge independently of the final stock requirement. These results refer to the most common models used in the literature.

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1 For the definitions see for example Dasgupta and Mitra (1999) or Nikaido (1968). In the latter term 'productive' is equivalent to 'workable'.

2 In Dutta (1993) it is assumed $\exists \beta > 0 \forall t : \|x\| > \beta$ and $y \in F_t(x)$ then $\|y\| < \|x\|$; Nermuth (1979) assumed $\forall t \forall x \in S : x \in F_t(x)$.

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