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Non-stationary Leontief-Walras economy

Abstract: An economy's stability, in the traditional sense, is permanently connected with its equilibrium state, since when we speak of a stable (in a local or global sense) economic system, we mean its ability to return to equlibrium after shocks. Such a meaning of stability is senseless in non-stationary economies, since they do not have any invariant states that are synonyms of the equilibria. By the example of Leontief-Walras model we shall show that non-stationarity of an economy does not exclude its stability, and equlibrium is not a sine qua non condition of stable growth.

Keywords: Walrasian system, economic equilibrium, stationary (non-stationary) equilibrium, stable growth.

JEL codes: D50, C62.

1. Introduction

In discordance with current trends in mathematical economics, in Leontief-Walras competitive market model we present below, a special role is played by distinction between different kinds of current inputs (for example raw materials, semifinished products, transportation services etc.) and production factors (different generations of capital and different kinds of work) in the production process. We assume that there are produced n different goods, that are supposed to be consumed or used in the production process (as inputs). At any moment some prescribed quantities of k production factors are needed in the process of production.

In Leontief-Walras economy production possibilities are unbounded. But at any moment production inputs are limited. The available quantity of production factors and output level (of consummable goods and production factors) depend on market prices. The prices change according to the classical market mechanism of equlibrating supply and demand.

A formal difference between our approach and its classical predecessor is as follows: we reject the assumption of stationarity which results in depriving the economy of a competitive equlibrium state (non-stationary dynamical systems do not have – in general – the so-called invariant states; see, for example, Panek (2005)).

2. Leontief-Walras model (A non-stationary version)

We assume that time is continuous and runs through $T = [0;\infty)$ half-axe. By $x(t) = (x_1(t), \dots, x_n(t))$ we denote the vector of goods produced at the moment t, $y(t) = (y_1(t), \dots, y_k(t))$ is the vector of production factors used at the moment t. Vector $p(t) = (p_1(t), \dots, p_n(t))^T$ stands for prices of goods produced at the moment t and $v(t) = (v_1(t), \dots, v_k(t))^T$ is the vector of production factors prices; upper index T of this vector denotes its transposition.

By $A(t)$ we denote a (n, n) non-negative matrix of production coefficients at t. Its element $a_{ij}(t)$ measures how much of good j is necessary to produce a unit of good i at t, so that to produce an output vector $x(t)$ production inputs $z(t) = xA(t)$ are needed.

$B(t)$ denotes a (n, k) non-negative matrix of production factors coefficients at time t. Its element $b_{ij}(t)$ denotes a necessary input of production factor j to produce a unit of good i at the moment t. Vector $y(t) = x(t)B(t)$ is an input of production factors which makes it possible to produce output $x(t)$ at t.

Behavior of consumers and producers in the economy is described by three vector-valued functions

$$\varphi : R_+^n \times R_+^k / \{0\} \times R_+ \to R_+^n,$$
$$\psi : R_+^n \times R_+^k / \{0\} \times R_+ \to R_+^k,$$
$$\zeta : R_+^n / \{0\} \times R_+ \to R_+^n,$$

whose values are demand for goods vector

$$\varphi(p, v, t) = \varphi_1(p_1, \dots, p_n, v_1, \dots, v_k, t), \dots, \varphi_n(p_1, \dots, p_n, v_1, \dots, v_k, t)),$$

factors supply vector

$$\psi(p, v, t) = (\psi_1(p_1, \dots, p_n; v_1, \dots, v_k, t), \dots, \psi_k(p_1, \dots, p_n; v_1, \dots, v_k, t))$$

at the moment t under goods prices $p = (p_1, \dots, p_n)^T$ and factors prices $v = (v_1, \dots, v_k)^T$, and average profit (at time t) vector (dependent on production level x and time t)

$$\xi(x, t) = (\xi_1(x_1, t), \dots, \xi_n(x_n, t))^T.$$

We assume (implicitly) that the global demand (supply) function is a sum of demand functions of individual consumers (supply functions of individual producers). In Leontief-Walras economy we are not interested in the behavior of a particular consumer (producer).

We assume that functions φ, ψ, ξ satisfy:

(I) $\varphi \in C^1\left(R_+^n \times R_+^k / \{0\} \times R_+\right)$ and

$$\forall p,v \geq 0, \forall \lambda > 0, \forall t \geq 0 \quad \left(\varphi\left(\lambda p, \lambda v, t\right) = \varphi\left(p, v, t\right)\right).$$

(II) $\psi \in C^1\left(R_+^n \times R_+^k / \{0\} \times R_+\right)$ and

$$\forall p,v \geq 0, \forall \lambda > 0, \forall t \geq 0 \quad \left(\psi\left(\lambda p, \lambda v, t\right) = \psi\left(p, v, t\right)\right).$$

(III) $\zeta \in C^1\left(R_+^n / \{0\} \times R_+\right)$ and

$$\forall i, \forall r > 0 \, \exists \bar{\zeta}_i^r \; \forall \begin{pmatrix} p \\ v \\ x^T \end{pmatrix} \in K_+^{2n+k}(r), \forall t \geq 0 \quad \left(\frac{\partial \zeta_i\left(x_i, t\right)}{\partial x_i} \geq \frac{\bar{\zeta}_i^r}{t+1} \right),$$

where $K_+^{2n+k}(r) := \left\{ w \in R_+^{2n+k} \,\middle|\, \|w\|_E \leq r \right\}$ denotes a closed ball of radius r centered at $0, \|\cdot\|_E$ is Euclid norm.

Positive homogeneity of degree 0 of the global demand and supply functions, φ, ψ respectively which is stated in conditions (I) and (II) state a well-known fact that demand and supply (production) are sensible to the structure of prices and not to the absolute level of prices. Condition (III) excludes decrease of the average profit of production of any good to zero-level, at given time t. We take the following assumptions on matrices $A(t)$, $B(t)$.

(IV) $A \in C^0\left(R_+\right)$ and

(a) $\forall t \geq 0$ each row of $A(t)$ contains a positive element,

(b) $\forall t, \tau \geq 0 \left(t \geq \tau \Rightarrow A(t) \leq A(\tau)\right)$,

(c) $A(0)$ is a productive matrix.

(V) $B \in C^0\left(R_+\right)$ and

(a) $\forall t \geq 0$ each column of $B(t)$ contains a positive element,

(b) $\forall t, \tau \geq 0 \left(t \geq \tau \Rightarrow B(t) \leq B(\tau)\right)$.

Conditions (IVa), (Va) state that production of any good needs inputs (of goods and production factors). Assumptions (IVa), (Vb) express that there is technological progress. Condition (IVc) says that the economy is capable to produce more goods than it uses as inputs at time $t = 0$. Together with condition (IVb) it ensures that the economy is always productive as a whole (at any moment $t \geq 0$ it is possible to deliver onto the market more goods than it is used up).

Walras' Law holds:

(VI) $\forall t \geq 0, \forall p, v, x \geq 0 \quad \left(\left\langle p, \varphi\left(p, v, t\right) \right\rangle = \left\langle v, \psi(p, v, t \right\rangle + \left\langle x, \zeta\,(x, t) \right\rangle \right)$.

According to this law, for any prices the value of global demand equals the value of global supply corrected by the average profit[1]. Price and production factor dy-

[1] It is possible to prove that Walras' Law holds. We do not do it clarity of presentation.

namics in horizon [0, ∞) are described by the following non-stationary system of differential equations[2]:

$$\dot{p}(t) = \sigma(t)\left[\varphi\left(p(t), v(t), t\right) - x(t)\left(E - A(t)\right)\right]^T, \tag{1}$$

$$\dot{v}(t) = \sigma(t)\left[x(t)B(t) - \psi\left(p(t), v(t), t\right)\right]^T, \tag{2}$$

$$\dot{x}^T(t) = \sigma(t)\left[p(t) - A(t)p(t) - B(t)v(t) - \zeta\left(x(t), t\right)\right], \tag{3}$$

where $\dot{p}(t) = \left(\dot{p}_1(t), \ldots, \dot{p}_n(t)\right)^T, \dot{v}(t) = \left(\dot{v}_1(t), \ldots, \dot{v}_k(t)\right)^T, \dot{x}(t) = \left(\dot{x}_1(t), \ldots, \dot{x}_n(t)\right)$, and $\sigma(t)$ is a positive coefficient at time t such that:

(VII) $\sigma \in C^0\left(R_+ \to R_+\right)$ and $\inf_{t\geq 0} = \underline{\sigma} > 0$.

The prices of goods and factors in (1) and (2) change in a proportion to excess demand. According to (3) production level varies depending on goods prices and costs. Positive relations (which allows to earn some extraordinary profits) between prices and costs encourage the producers to increase production; otherwise production shrinks.

Interpretation of (VII) is similar to interpretation of (III). Existence of a positive lower bound of function σ stated in (III) assures that prices react to changes in supply or demand (equations (2) and (3)) and production is sensitive to prices and costs (eq. (3))[3].

We assume that system (1)-(3) has a unique solution on half-axis $T = [0, \infty)$ under initial condition[4]

$$p(0) = p^0 > v(0) = v^0 > 0, x(0) = x^0 > 0. \tag{4}$$

Definition 1. *We call a solution of system (1)-(3) in* $T = [0, \infty)$ *under initial condition (4) a* $\left(p^0, v^0, x^0, \infty\right)$ *a feasible growth process in non-stationary Leontief-Walras economy. Functions:* p_T, v_T, x_T *we call a* (p^0, ∞) *– feasible trajectory of goods prices,* (v^0, ∞) *– feasible trajectory of factors prices and* (x^0, ∞) *– feasible trajectory of production, respectively.*

Non-stationary Leontief-Walras economy is never in equlibrium in the classical sense, for there is no – in general – such a particular solution $\left(\bar{p}, \bar{v}, \bar{x}\right)_T$ of (1)-(4)

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[2] Compare this system to its stationary version presented in Morishima (1964) or Panek (2000).

[3] The scalar function $\sigma(t)$ in equations (1), (3) could be replaced by diagonal functional matrices $\sigma^1(t) = \sigma(t)diag\left(\sigma_1^1, \ldots, \sigma_n^1\right)$, $\sigma^3(t) = \sigma(t)diag\left(\sigma_1^3, \ldots, \sigma_n^3\right)$, respectively, and in equation (2) we could put $\sigma^2(t) = \sigma(t)diag\left(\sigma_1^2, \ldots, \sigma_n^2\right)$ with positive elements on the diagonal, see a comment following after lemma 1.

[4] Under assumptions (I)-(VII) system (1)-(3) has an unique solution on interval $[0, t_1)$, $t_1 \leq \infty$, such that for $t = 0$ condition (4) holds. We assume a little bit more, namely, that the solution is welldefined on the whole half-axis $[0, \infty)$.

that for all moments $t \geq 0$ it holds $\overline{p}(t) = \overline{p} = \text{const} > 0, \overline{v}(t) = \overline{v} = \text{const} > 0, \overline{x}(t) =$ $= \overline{x} = \text{const} > 0$ and:

$$\begin{gathered} \varphi(\overline{p},\overline{v},t) - \overline{x}(E - A(t)) = 0 \\ \overline{x}B(t) - \psi(\overline{p},\overline{v},t) = 0 \\ (E - A(t))\overline{p} - B(t)\overline{v} - \zeta(\overline{x},t) = 0. \end{gathered} \tag{5}$$

Nevertheless, the feasible processes have some interesting asymptotical properties, for $t \to \infty$, that we consider below. At the end of the point we shall formulate and prove a simple lemma places feasible processes on a $(2n + k - 1)$-dimensional sphere whose radius is determined by the initial state of economy $\left(\text{triple: } (p^0, v^0, x^0)\right)$.

Lemma 1. *Under assumption (VI) (Walras' Law) every* (p^0, v^0, x^0, ∞)*-feasible growth process is contained in a* $(2n + k)$*-dimensional sphere centered at* 0 *with radius* $r = \left(\langle p^0, p^0\rangle + \langle v^0, v^0\rangle + \langle x^0, x^0\rangle\right)^{1/2}$.

Proof. Fix some (p^0, v^0, x^0, ∞)-feasible growth process $(p, v, x)_T$. Then $\forall t \geq 0$:

$$\begin{aligned} &\langle p(t), \dot{p}(t)\rangle + \langle v(t), \dot{v}(t)\rangle + \langle x(t), \dot{x}(t)\rangle = \\ &= \left\langle p(t), \sigma(t)\left[\varphi(p(t), v(t), t) - x(t)(E - A(t))\right]\right\rangle + \\ &+ \left\langle v(t), \sigma(t)\left[x(t)B(t) - \psi(p(t), v(t), t)\right]\right\rangle + \\ &+ \left\langle x(t), \sigma(t)\left[(E - A(t))p(t) - B(t)v(t) - \zeta(x(t), t)\right]\right\rangle = \\ &= \sigma(t)\Big[\left\langle p(t), \varphi(p(t), v(t), t)\right\rangle - \left\langle x(t), (E - A(t))p(t)\right\rangle + \\ &+ \left\langle x(t), B(t)v(t)\right\rangle - \left\langle x(t), \zeta(x(t), t)\right\rangle\Big] = 0, \end{aligned}$$

so that

$$\begin{aligned} &\int_0^t \left[\langle p(t), \dot{p}(t)\rangle + \langle v(t), \dot{v}(t)\rangle + \langle x(t), \dot{x}(t)\rangle\right] dt = \\ &= \frac{1}{2}\left[\langle p(t), p(t)\rangle + \langle v(t), v(t)\rangle + \langle x(t), x(t)\rangle - \langle p^0, p^0\rangle - \langle v^0, v^0\rangle - \langle x^0, x^0\rangle\right] = 0. \end{aligned}$$

Therefore $\forall t \geq 0$

$$\sum_{i=1}^n p_i^2(t) + \sum_{j=1}^k v_i^2(t) + \sum_{i=1}^n x_i^2(t) = r,$$

where $r := \left(\langle p^0, p^0\rangle + \langle v^0, v^0\rangle + \langle x^0, x^0\rangle\right)^{1/2}$. □

It is not difficult to check that if we put in equations (1),(3) some diagonal functional matrices $\sigma^i(t) = \sigma(t)diag(\sigma_1^i, \ldots, \sigma_n^i)$, $i = 1, 3$ instead of *scalar* function $\sigma(t)$ and $\sigma^2(t) = \sigma(t)diag(\sigma_1^2, \ldots, \sigma_n^2)$ with positive elements on the diagonal, then under (VI) any (p^0, v^0, x^0, ∞)-feasible growth process would be located in boundary of an $(2n + k)$-dimensional elipsoid centered at 0, with radii d_j, $j = 1, \ldots, 2n + k$, defined as:

$$d_j := \sqrt{d\sigma_i^1} \text{ for } j = 1, \ldots, n,$$

$$d_{n+j} := \sqrt{d\sigma_i^2} \text{ for } j = 1, \ldots, k,$$

$$d_{n+k+j} := \sqrt{d\sigma_i^3} \text{ for } j = 1, \ldots, n,$$

where

$$d := \sum_{i=1}^{n}\left[\left(p_i^0\right)^2 / \sigma_i^1\right] + \sum_{j=1}^{k}\left[\left(v_j^0\right)^2 / \sigma_i^2\right] + \sum_{i=1}^{n}\left[\left(x_i^0\right)^2 / \sigma_i^3\right].$$

3. Local stability

If, in Leontief-Walras economy, one could point at a state of equlibrium $\bar{p}, \bar{v}, \bar{x}$ satisfying on $T = [0, \infty)$ conditions (5), then stability (local or global) of the economy would imply convergence (local or global) of feasible processes toward equilibrium. Since such an equilibrium state – as we asserted – does not exist, then instead of classical covergence of (p^0, v^0, x^0, ∞)-feasible growth processes toward a state of equilibrium (toward equilibrium prices of goods $\bar{p}$ and factors $\bar{v}$ and production level $\bar{x}$, respectively) we shall investigate convergence of processes toward each other, when they start from different states. The below definition mimics definition 2.3.

Definition 2. *Non-stationary Leontief-Walras economy is called locally layer-wise asymptotically stable, when for any pair of* $\left(p^{0i}, v^{0i}, x^{0i}, \infty\right)$*-feasible growth processes,* $i = 1, 2$ *it holds*

$$\left\| \begin{matrix} p^{01} \\ v^{01} \\ x^{01^T} \end{matrix} \right\|_E = \left\| \begin{matrix} p^{02} \\ v^{02} \\ x^{02^T} \end{matrix} \right\|_E \Rightarrow \lim_t \left\| \begin{matrix} p^1(t) - p^2(t) \\ v^1(t) - v^2(t) \\ x^1(t)^T - x^2(t)^T \end{matrix} \right\|_E = 0.$$

Let us denote by

$$J(p, v, t) = \begin{pmatrix} \dfrac{\partial \varphi(p, v, t)}{\partial p} & \dfrac{\partial \varphi(p, v, t)}{\partial v} \\ -\dfrac{\partial \psi(p, v, t)}{\partial p} & -\dfrac{\partial \psi(p, v, t)}{\partial v} \end{pmatrix}_{(n+k, n+k)}. \tag{6}$$

Jacobi's functional matrix of pair of vector-valued functions $\varphi(p, v, t)$ and $\psi(p, v, t)$ of variables $p \in R_+^n, v \in R_+^k, t \in R_+$. We shall assume:

(VIII) $\forall r > 0, \forall d > 0 \exists \varepsilon > 0, \forall p^1, v^1, x^1 > 0, \forall p^2, v^2, x^2 > 0$
$\forall p \in \left[p^1, p^2\right], \forall v \in \left[v^1, v^2\right], \forall t \geq 0$

$$\left(\left\| \begin{matrix} p^1 \\ v^1 \\ x^{1^T} \end{matrix} \right\|_E = \left\| \begin{matrix} p^2 \\ v^2 \\ x^{2^T} \end{matrix} \right\|_E = r \wedge \left(\left\| p^1 - p^2 \right\|_E \geq d \vee \left\| v^1 - v^2 \right\|_E \geq d \right) \right) \Rightarrow$$
$$\Rightarrow \left(\left(\left(p^2 - p^1\right)^T, \left(v^2 - v^1\right)^T \right) J(p, v, t) \begin{pmatrix} p^2 - p^1 \\ v^2 - v^1 \end{pmatrix} < -\frac{\varepsilon}{t+1} \right).$$

Under this assumption

$$\frac{\partial \varphi(p, v, t)}{\partial p_i} < 0 \text{ for } i = 1, \ldots, n$$

and

$$\frac{\partial \psi(p, v, t)}{\partial v_j} < 0 \text{ for } j = 1, \ldots, k,$$

i.e. demand for goods decreases, and factor supply increases when prices grow. So that we have to deal with a "normal" economy, where there are valid natural (classical) reactions of agents facing changes in prices of goods and factors.

Lemma 2. *If Jacobi's functional matrix (6) satisfies condition (VIII) then*
$\forall r > 0, \forall d > 0 \exists \varepsilon > 0, \forall p^1, v^1, x^1 > 0, \forall p^2, v^2, x^2 > 0$

$$\left(\left\| \begin{matrix} p^1 \\ v^1 \\ x^{1^T} \end{matrix} \right\|_E = \left\| \begin{matrix} p^2 \\ v^2 \\ x^{2^T} \end{matrix} \right\|_E = r \wedge \left(\left\| p^1 - p^2 \right\|_E \geq d \vee \left\| v^1 - v^2 \right\|_E \geq d \right) \right) \Rightarrow$$
$$\Rightarrow \left\langle p^1, \varphi\left(p^2, v^2, t\right) \right\rangle - \left\langle v^1, \psi\left(p^2, v^2, t\right) \right\rangle + \left\langle p^2, \varphi\left(p^1, v^1, t\right) \right\rangle - \left\langle v^2, \psi\left(p^1, v^1, t\right) \right\rangle > \frac{\varepsilon}{t+1}.$$

Proof. Fix any numbers $r > 0$, $d > 0$. Let us choose such vectors p^i, v^i, $x^i > 0$, $i = 1, 2$ satisfying

$$\left\| \begin{matrix} p^1 \\ v^1 \\ x^{1^T} \end{matrix} \right\|_E = \left\| \begin{matrix} p^2 \\ v^2 \\ x^{2^T} \end{matrix} \right\|_E = r \wedge \left(\left\| p^1 - p^2 \right\|_E \geq d \vee \left\| v^1 - v^2 \right\|_E \geq d \right).$$

Denote

$$p(\tau) = p^1 + \left(p^2 - p^1\right)\tau,$$
$$v(\tau) = v^1 + \left(v^2 - v^1\right)\tau,$$

and (for any fixed $t \geq 0$):

$$\phi(\tau,t) = \left\langle p^2 - p^1, \varphi\left(p(\tau), v(\tau), t\right) - \varphi\left(p^1, v^1, t\right)\right\rangle -$$
$$-\left\langle v^2 - v^1, \psi\left(p(\tau), v(\tau), t\right) - \psi\left(p^1, v^1, t\right)\right\rangle,$$

where $\tau \in [0,1]$. Then $\forall t \geq 0 \left(\phi(\cdot,t) \in C^1[0,1]\right)$ and

$$p(0) = p^1, p(1) = p^2, v(0) = v^1, v(1) = v^2, \phi(0,t) = 0$$

$$\frac{\partial \phi(\tau,t)}{\partial \tau} = \lambda^T J\left(p(\tau), v(\tau), t\right)\lambda\ ,$$

where $\lambda = \begin{pmatrix} p^2 - p^1 \\ v^2 - v^1 \end{pmatrix}$. According to assumption (VIII) there exists a number ε such that

$$\forall \tau \in [0,1], \forall t \geq 0 \left(\lambda^T J\left(p(\tau), v(\tau), t\right)\lambda < -\frac{\varepsilon}{t+1}\right).$$

Since $\phi(0,t) = 0, \dfrac{\partial \phi(\tau,t)}{\partial \tau} < -\dfrac{\varepsilon}{t+1}$ then

$$\phi(1,t) = \left\langle p^2 - p^1, \varphi\left(p^2, v^2, t\right) - \varphi\left(p^1, v^1, t\right)\right\rangle -$$
$$-\left\langle v^2 - v^1, \psi\left(p^2, v^2, t\right) - \psi\left(p^1, v^1, t\right)\right\rangle < -\frac{\varepsilon}{t+1}.$$

The proof is completed because it follows from Walras' Law (VI) that

$$\left\langle p^i, \varphi\left(p^i, v^i, t\right)\right\rangle - \left\langle v^i, \psi\left(p^i, v^i, t\right)\right\rangle = \left\langle x, \zeta(x,t)\right\rangle, i = 1,2.$$

□

The condition of lengths equality of initial vectors

$$w^1 = \begin{pmatrix} p^1 \\ v^1 \\ x^{1^T} \end{pmatrix},\ w^2 = \begin{pmatrix} p^2 \\ v^2 \\ x^{2^T} \end{pmatrix} \tag{7}$$

in assumption (VIII) is obvious with respect to lemma 1: only processes contained in the very same $(2n + k)$-dimensional sphere can converge to each other. Processes for which initial states w^1, w^2 do not belong to the same sphere will never reach a distance smaller than $\left|r^1 - r^2\right| > 0$, where $r^i = \left\|w^i\right\|_E, i = 1,2.$

In fact, theorem 1 asserts that in non-stationary Leontief-Walras economy growth processes converge to each other. While proving it, the following lemma plays an important role:

Lemma 3. *Under all assumptions*
$\forall r > 0, \forall d > 0 \exists \varepsilon > 0 \forall p^1, v^1, x^1 > 0, \forall p^2, v^2, x^2 > 0, \forall t \geq 0$

$$\begin{aligned}
&\left\|w^1\right\|_E = \left\|w^2\right\|_E = r \wedge \left(\left\|p^1 - p^2\right\|_E \geq d \vee \left\|v^1 - v^2\right\|_E \geq d \vee \left\|x^1 - x^2\right\|_E \geq d\right) \Rightarrow \\
&\Rightarrow \left\langle p^1, \varphi\left(p^2, v^2, t\right)\right\rangle - \left\langle v^1, \psi\left(p^2, v^2, t\right)\right\rangle + \\
&+ \left\langle p^2, \varphi\left(p^1, v^1, t\right)\right\rangle - \left\langle v^2, \psi\left(p^1, v^1, t\right)\right\rangle + \\
&+ \left\langle x^1, \zeta\left(x^2, t\right)\right\rangle - \left\langle x^2, \zeta\left(x^1, t\right)\right\rangle > \frac{\varepsilon}{t+1},
\end{aligned}$$

where vectors w^1, w^2 *are of form (7).*

Proof. The proof of this lemma is similar to proof of lemma 2. For all $\tau \in [0,1]$ denote:

$$\begin{aligned}
p(\tau) &= p^1 + \left(p^2 - p^1\right)\tau, \\
v(\tau) &= v^1 + \left(v^2 - v^1\right)\tau, \\
x(\tau) &= x^1 + \left(x^2 - x^1\right)\tau,
\end{aligned}$$

and

$$\begin{aligned}
\Gamma(\tau, t) &= \left\langle p^2 - p^1, \varphi\left(p(\tau), v(\tau), t\right) - \varphi\left(p^1, v^1, t\right)\right\rangle - \\
&- \left\langle v^2 - v^1, \psi\left(p(\tau), v(\tau), t\right) - \psi\left(p^1, v^1, t\right)\right\rangle - \left\langle x^2 - x^1, \zeta\left(x(\tau), t\right) - \zeta\left(x^1, t\right)\right\rangle.
\end{aligned}$$

Then $\forall t \geq 0 \left(\Gamma(\cdot, t) \in C^1[0,1], \Gamma(0,t) = 0\right)$ and

$$\frac{\partial \Gamma(\tau, t)}{\partial \tau} = \gamma^T J\left(p(\tau), v(\tau), x(\tau), t\right)\gamma,$$

where

$$\gamma = \begin{pmatrix} p^2 - p^1 \\ v^2 - v^1 \\ x^{2^T} - x^{1^T} \end{pmatrix},$$

and

$$J(p,v,x,t)=\begin{pmatrix} \dfrac{\partial\varphi(p,v,t)}{\partial p} & \dfrac{\partial\varphi(p,v,t)}{\partial v} & -\left(E-A(t)\right)^{T} \\ \dfrac{-\partial\psi(p,v,t)}{\partial p} & -\dfrac{\partial\psi(p,v,t)}{\partial v} & B(t)^{T} \\ E-A(t) & -B(t) & -\dfrac{\partial\zeta(x,t)}{\partial x} \end{pmatrix}_{(2n+k,2n+k)}$$

is Jacobi's functional matrix of the right-hand-side of system (1)-(3). Here and on

$$\frac{\zeta(x,t)}{\partial x}=\begin{pmatrix} \dfrac{\zeta_1(x_1,t)}{\partial x_1} & & 0 \\ & \ddots & \\ 0 & & \dfrac{\zeta_n(x_n,t)}{\partial x_n} \end{pmatrix}.$$

It can be easily seen that

$$\frac{\partial\Gamma(\tau,t)}{\partial\tau}=\lambda^{T}J\left(p(\tau),v(\tau),t\right)\lambda-\left(x^{2}-x^{1}\right)\frac{\zeta(x,t)}{\partial x}\Big|_{x=x(\tau)}\left(x^{2}-x^{1}\right)^{T},$$

where functional matrix $J(p, v, t)$ is given by (6) and $\lambda=\begin{pmatrix} p^{2}-p^{1} \\ v^{2}-v^{1} \end{pmatrix}$.

Fix any numbers $r > 0$, $d > 0$. Under (VIII) there exists such a number $\varepsilon' > 0$ that $\forall p^{1},v^{1},x^{1}>0,\forall p^{2},v^{2},x^{2}>0,\forall t\geq 0$
from equalities

$$\left\|w^{1}\right\|_{E}=\left\|\begin{matrix} p^{1} \\ v^{1} \\ x^{1^{T}} \end{matrix}\right\|_{E}=\left\|w^{2}\right\|_{E}=\left\|\begin{matrix} p^{2} \\ v^{2} \\ x^{2^{T}} \end{matrix}\right\|_{E}=r$$

and condition

$$\left\|p^{1}-p^{2}\right\|_{E}\geq d \text{ or } \left\|v^{1}-v^{2}\right\|_{E}\geq d,$$

it follows

$$\lambda^{T}J\left(p(\tau),v(\tau),t\right)\lambda<-\frac{\varepsilon'}{t+1}. \quad (*)$$

Condition (*) holds for $\forall\tau\in[0,1],\forall t\geq 0$. Under assumption (III)

$$\forall w \in K_+^{2n+k}(r), \forall t \geq 0, \forall i \left(\frac{\partial \zeta_i(x_i,t)}{\partial x_i} \geq \frac{\bar{\zeta}_i^r}{t+1} > 0 \right),$$

where $w = \begin{pmatrix} p \\ v \\ x^T \end{pmatrix}$. So that $\exists \varepsilon'' > 0\ \forall w^1, w^2 \in K_+^{2n+k}(r),\ \forall \tau \in [0,1]$

$$\left\| x^1 - x^2 \right\|_E \geq d \Rightarrow$$

$$\Rightarrow \left(x^2 - x^1\right) \frac{\zeta(x,t)}{\partial x}\bigg|_{x=x(\tau)} \left(x^2 - x^1\right)^T \geq \frac{1}{t+1}\left(x^2 - x^1\right)\bar{\zeta}^r \left(x^2 - x^1\right)^T > \frac{\varepsilon''}{t+1}, (**)$$

where

$$w^1 = \begin{pmatrix} p^1 \\ v^1 \\ x^{1^T} \end{pmatrix},\ w^2 = \begin{pmatrix} p^2 \\ v^2 \\ x^{2^T} \end{pmatrix},\ \bar{\zeta}^r = \begin{pmatrix} \bar{\zeta}^1 & & 0 \\ & \ddots & \\ 0 & & \bar{\zeta}^n \end{pmatrix}.$$

Let $\varepsilon = \min\{\varepsilon', \varepsilon''\}$. From (*), (**) it follows:

$$\forall \tau \in [0,1], \forall t \geq 0 \left(\frac{\partial \Gamma(\tau,t)}{\partial \tau} < -\frac{\varepsilon}{t+1} \right),$$

if $\left\| p^1 - p^2 \right\|_E \geq d$ or $\left\| v^1 - v^2 \right\|_E \geq d$ or $\left\| x^1 - x^2 \right\|_E \geq d$.

Since $\Gamma(0,t) = 0$ then

$$\Gamma(1,t) = \left\langle p^2 - p^1, \varphi\left(p^2, v^2, t\right) - \varphi\left(p^1, v^1, t\right)\right\rangle - \left\langle v^2 - v^1, \psi\left(p^2, v^2, t\right) - \right.$$

$$\left. - \psi\left(p^1, v^1, t\right)\right\rangle - \left\langle x^2 - x^1, \zeta\left(x^2, t\right) - \zeta\left(x^1, t\right)\right\rangle < -\frac{\varepsilon}{t+1}.$$

Further on, from Walras' Law – after some simple transformations – the thesis follows. □

Theorem 1. *Under conditions (I)-(VII) non-stationary Leontief-Walras economy is layerwise asymptotically stable.*

Proof. Fix a pair of $\left(p^{0i}, v^{0i}, x^{0i}, \infty\right)$-feasible growth processes, $i = 1, 2$, such that

$$\left\| \begin{matrix} p^{01} \\ v^{01} \\ x^{01^T} \end{matrix} \right\|_E = \left\| \begin{matrix} p^{02} \\ v^{02} \\ x^{02^T} \end{matrix} \right\|_E = r > 0.$$

From lemma 1 it follows that the processes are contained in a sphere with radius r centered at 0. If $p^{01} = p^{02}, v^{01} = v^{02}$ and $x^{01} = x^{02}$ then

$$\forall t \geq 0 \left(p^1(t) = p^2(t), v^1(t) = v^2(t), x^1(t) = x^2(t) \right).$$

If, on the other hand $p^{01} \neq p^{02} \vee v^{01} \neq v^{02} \vee x^{01} \neq x^{02}$, then

$$\forall t \geq 0 \left(p^1(t) \neq p^2(t) \vee v^1(t) \neq v^2(t) \vee x^1(t) \neq x^2(t) \right).$$

Define

$$V(t) := \frac{1}{2}\Big[\left\langle p^2(t) - p^1(t), p^2(t) - p^1(t) \right\rangle + \left\langle v^2(t) - v^1(t), v^2(t) - v^1(t) \right\rangle +$$

$$+ \left\langle x^2(t) - x^1(t), x^2(t) - x^1(t) \right\rangle \Big] =$$

$$= \frac{1}{2}\left(\left\| p^2(t) - p^1(t) \right\|_E^2 + \left\| v^2(t) - v^1(t) \right\|_E^2 + \left\| x^2(t) - x^1(t) \right\|_E^2 \right).$$

Of course $V \in C^1\left([0, \infty) \to R_+ \right)$ and

$$\dot{V}(t) = \sigma(t)\Big[\left\langle v^1(t), \psi\left(p^2(t), v^2(t), t \right) \right\rangle + \left\langle v^2(t), \psi\left(p^1(t), v^1(t), t \right) \right\rangle +$$

$$+ \left\langle x^1(t), \zeta\left(x^2(t), t \right) \right\rangle + \left\langle x^2(t), \zeta\left(x^1(t), t \right) \right\rangle -$$

$$- \left\langle p^1(t), \varphi\left(p^2(t), v^2(t), t \right) \right\rangle - \left\langle p^2(t), \varphi\left(p^1(t), v^1(t), t \right) \right\rangle \Big] < 0.$$

Suppose that there exists such a number $d > 0$ that

$$\forall t \geq 0 \left(\left\| p^2(t) - p^1(t) \right\|_E^2 \geq d \vee \left\| v^2(t) - v^1(t) \right\|_E^2 \geq d \vee \left\| x^2(t) - x^1(t) \right\|_E^2 \geq d \right).$$

Then, under assumptions (I)-(VIII), in the light of lemma 3, there exists a number $\varepsilon > 0$

$$\forall t \geq 0 \left(\dot{V}(t) < -\underline{\sigma} \frac{\varepsilon}{t+1}, \right.$$

so that $0 \leq V(t) < V(0) - \underline{\sigma}\varepsilon \ln(t+1) \to -\infty$, when $t \to \infty$, which is impossible. The thesis has been proven. □

4. Global stability of non-stationary Leontief-Walras economy with relative prices

Going from absolute prices of goods and factors $p = \left(p_1, \ldots, p_n \right)^T, v = \left(v_1, \ldots, v_n \right)^T$ to a relative prices system, wherein prices of all goods and factors are expressed in terms of k-th factor's unit $\left(v_k = 1 \right)$ and denoting

$$\hat{v}(t) = (v_1(t), \ldots, v_{k-1}(t)),$$

$$\hat{\psi}(p, \hat{v}, t) = (\psi_1(p_1, \ldots, p_n, v_1, \ldots, v_{k-1}, 1, t), \ldots, \psi_n(p_1, \ldots, p_n, v_1, \ldots, v_{k-1}, 1, t))$$

$$\hat{B}(t) = \begin{pmatrix} b_{11}(t) & \ldots & b_{1k-1}(t) \\ b_{21}(t) & \ldots & b_{2k-1}(t) \\ \vdots & & \\ b_{n1}(t) & \ldots & b_{nk-1}(t) \end{pmatrix}$$

we replace dynamic system (1)-(3) of $2n + k$ equations with the following $2n + k - 1$ equations:

$$\dot{p}(t) = \sigma(t)\left[\varphi(p(t), \hat{v}(t), 1, t) - x(t)(E - A(t))\right]^T, \tag{8}$$

$$\dot{v}(t) = \sigma(t)\left[x(t)\hat{B}(t) - \hat{\psi}(p(t), \hat{v}(t), 1, t)\right]^T, \tag{9}$$

$$\dot{x}^T(t) = \sigma(t)\left[p(t) - A(t)p(t) - \hat{B}(t)\hat{v}(t) - b^k(t) - \zeta(x(t), t)\right], \tag{10}$$

where $b^k(t)$ stands for k-th column of matrix $B(t)$ from (1)-(3), so that $B(t) = \left[\hat{B}(t), b^k(t)\right]$. Similarily, as in definition 1, a positive solution of system (8)-(10), defined on $[0, \infty)$ under initial condition

$$p(0) = p^0 > 0, \hat{v}(0) = \hat{v}^0 > 0, x(0) = x^0 > 0$$

is called $(p^0, \hat{v}^0, x^0, \infty)$-feasible growth processes in a non-stationary Leontief-Walras economy with relative prices and is denoted by $(p, \hat{v}, x)_T$.

Definition 3. *Non-stationary Leontief-Walras economy with relative prices is called globally asymptotically stable, when* $\forall p^{0i} > 0, \hat{v}^{0i} > 0, x^{0i} > 0$ *and any pair of* $(p^{0i}, v^{0i}, x^{0i}, \infty)$*-feasible growth processes,* $i = 1, 2$*, satisfies*

$$\left\|p^1(t) - p^2(t)\right\|_E \xrightarrow{t} 0,$$

$$\left\|\hat{v}^1(t) - \hat{v}^2(t)\right\|_E \xrightarrow{t} 0,$$

$$\left\|x^1(t) - x^2(t)\right\|_E \xrightarrow{t} 0.$$

An equivalent of assumption (VIII) is now (VIII')

(VIII') $\forall d > 0\ \exists \varepsilon > 0\ \forall p^1, \hat{v}^1, x^1 > 0\ \forall p^2, v^2, x^2 > 0\ \forall p \in \left[p^1, p^2\right], \forall \hat{v} \in \left[\hat{v}^1, \hat{v}^2\right], \forall t \geq 0$
$\left(\left\|p^1 - p^2\right\|_E \geq d \vee \left\|\hat{v}^1 - \hat{v}^2\right\|_E \geq d \Rightarrow \lambda^T J(p, \hat{v}, t)\lambda < -\dfrac{\varepsilon}{t+1}\right)$, where

$$J(p,\hat{v},t)=\begin{pmatrix} \dfrac{\partial\varphi(p,\hat{v},1,t)}{\partial p} & \dfrac{\partial\varphi(p,\hat{v},1,t)}{\partial\hat{v}} \\ -\dfrac{\partial\hat{\psi}(p,\hat{v},1,t)}{\partial p} & \dfrac{\partial\hat{\psi}(p,\hat{v},1,t)}{\partial\hat{v}} \end{pmatrix}_{(n+k-1,n+k-1)}, \quad \lambda=\begin{pmatrix} p^2-p^1 \\ \hat{v}^2-\hat{v}^1 \end{pmatrix}.$$

Interpretation of (VIII') is similar to interpretation of (VIII).

Lemma 4. *Under assumptions (I)-(VII), (VIII')*

$$\forall d'>0\exists\varepsilon>0\forall p^1,\hat{v}^1,x^1>0,\forall p^2,\hat{v}^2,x^2>0,$$
$$\forall t\geq 0\Big(\left\|p^1-p^2\right\|_E\geq d\vee\left\|\hat{v}^1-\hat{v}^2\right\|_E\geq d\vee\left\|x^1-x^2\right\|_E\geq d\Rightarrow$$
$$\Rightarrow\left\langle p^1,\varphi\left(p^2,\hat{v}^2,1,t\right)\right\rangle-\left\langle\hat{v}^1,\hat{\psi}\left(p^2,\hat{v}^2,1,t\right)\right\rangle+\left\langle p^2,\varphi\left(p^1,\hat{v}^1,1,t\right)\right\rangle-$$
$$-\left\langle\hat{v}^2,\hat{\psi}\left(p^1,\hat{v}^1,1,t\right)\right\rangle-\left\langle x^1,\zeta\left(x^2,t\right)\right\rangle-\left\langle x^2,\zeta\left(x^1,t\right)\right\rangle>\frac{\varepsilon}{t+1}\Big).$$

We omit proof of this lemma, since it runs in a way analogous to proofs of lemmas 2, 3. By lemma 4 and using proof of theorem 1 we can easily prove theorem 2.

Theorem 2. *Under assumptions (I)-(VII), (VIII') non-stationary Leontief-Walras economy is globally asymptotically stable.*

Proof. Fix any two feasible growth processes $\left(p^1,\hat{v}^1,x^1\right)_T,\left(p^2,\hat{v}^2,x^2\right)_T$ satisfying (respectively) initial conditions

$$p^i(0)=p^{0i}>0,$$
$$\hat{v}^{0i}=\hat{v}^{0i}>0,$$
$$x^i(0)=x^{0i}>0,$$

$\left(i=1,2;T=[0,\infty)\right)$. If $p^{01}=p^{02}$, $\hat{v}^{01}=\hat{v}^{02}$ and $p^{01}=p^{02}$ then

$$\forall t\geq 0\left(p^1(t)=p^2(t),v^1(t)=v^2(t),x^1(t)=x^2(t)\right).$$

Suppose that $p^{01}\neq p^{02}\vee v^{01}\neq v^{02}\vee x^{01}\neq x^{02}$. Then

$$\forall t\geq 0\left(p^1(t)\neq p^2(t)\vee v^1(t)\neq v^2(t)\vee x^1(t)\neq x^2(t)\right).$$

Define

$$V(t):=\frac{1}{2}\Big[\left\langle p^2(t)-p^1(t),p^2(t)-p^1(t)\right\rangle+\left\langle\hat{v}^2(t)-\hat{v}^1(t),\hat{v}^2(t)-\hat{v}^1(t)\right\rangle+$$
$$+\left\langle x^2(t)-x^1(t),x^2(t)-x^1(t)\right\rangle\Big].$$

Under the assumptions $V \in C^1(R_+)$ and

$$\begin{aligned}\dot{V}(t) = \sigma(t)\Big[&\big\langle \hat{v}^1(t), \hat{\psi}\left(p^2(t), \hat{v}^2(t), t\right)\big\rangle + \big\langle \hat{v}^2(t), \hat{\psi}\left(p^1(t), \hat{v}^1(t), t\right)\big\rangle + \\ &+ \big\langle x^1(t), \zeta\left(x^2(t), t\right)\big\rangle + \big\langle x^2(t), \zeta\left(x^1(t), t\right)\big\rangle - \\ &- \big\langle p^1(t), \varphi\left(p^2(t), \hat{v}^2(t), t\right)\big\rangle - \big\langle p^2(t), \varphi\left(p^1(t), \hat{v}^1(t), t\right)\big\rangle\Big] < 0.\end{aligned}$$

If the economy is not globally asymptotically stable, then there exist two feasible processes $\left(p^1, \hat{v}^1, x^1\right)_T, \left(p^2, \hat{v}^2, x^2\right)_T$ such that

$$\forall t \geq 0\left(\left\|p^2(t) - p^1(t)\right\|_E^2 \geq d \vee \left\|v^2(t) - v^1(t)\right\|_E^2 \geq d \vee \left\|x^2(t) - x^1(t)\right\|_E^2 \geq d\right),$$

for a number $d > 0$. Then, by assumptions (VII), (VIII'), there exists a number $\varepsilon > 0$ such that

$$\forall t \geq 0\left(\dot{V}(t) < -\underline{\sigma}\frac{\varepsilon}{t+1}\right),$$

and therefore $\lim_t V(t) = -\infty$ – contradiction. □

5. Conclusions

In the related literature, stability notion is strongly identified with equilibrium. The paper shows that stable growth of a non-stationary economy – which in fact we deal with in reality – does not demand convergence toward an equilibrium state. Moreover, "equilibrium problem" ceases to exist in its traditional sense, which does not preclude examining stability properties of competitive economy functioning under classical market mechanism.

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