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Local stability of the competitive economy with relative prices and discrete time

Abstract: The paper presents a simple proof of a local asymptotical stability of the non-linear stationary Walrasian system with discrete time and relative prices. The proof uses a notion of global asymptotical stability in zero linear approximation of the non-linear system under stationary conditions.
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1. Walrasian system with relative prices

Stationary Walrasian system in a continuous version is a system of differential equations of the dynamics of prices in the n-products economy [Hansen (1976), Intriligator (1971), Karlin (1959), McKenzie (2002), Takayama (1985)]:

$$\dot{p}(t) = \sigma f\left(p(t)\right), \tag{1}$$

where
t – variable continuous of time, $t \in R_{+}^{1} = [0, +\infty)$,
n – number of goods (produced and/or used),
$p(t) = \left(p_1(t), \ldots, p_n(t)\right)$ – vector of goods prices at moment t,
$\dot{p}(t) = \left(\dot{p}_1(t), \ldots, \dot{p}_n(t)\right)$ –derivative trajectory of prices at moment t,
$f\left(p(t)\right) = \left(f_1\left(p(t)\right), \ldots, f_n\left(p(t)\right)\right)$ – function of the excess demand for goods at moment t,

$$f(p(t)) = f^{d}(p(t)) - f^{s}(p(t)),$$

$f^{d}\left(p(t)\right) = \left(f_1^{d}\left(p(t)\right), \ldots, f_n^{d}\left(p(t)\right)\right)$ – function of the aggregated demand for goods at moment t,

$f^s(p(t)) = (f_1^s(p(t)),...,f_n^s(p(t)))$ – function of the aggregated supply of goods at moment t,
σ – positive rate (index) of proportionality (coefficient of prices reaction to changes in demand and/or supply).

Demand and supply of goods depend on current prices; on the other hand, they determine the directions and speed of their changes in the future. If we replace in (1) derivative $\dot{p}(t)$ by increase of $\Delta p(t) = p(t+1) - p(t)$ and assume that the time variable t is an integer, t = 0,1,..., then we have a discrete equivalent to the system (1):

$$\Delta p(t) = \sigma f(p(t)) \tag{2}$$

which in literature is often presented in an equivalent recursive form

$$p(t+1) = \varphi(p(t)), \tag{3}$$

where

$$\varphi(p(t)) = p(t) + \sigma f(p(t)).$$

We assume that the vector function of surplus demand $f(p)$, we assume that [8, 9]:
(i) $f \in C^1(R_+^n \setminus \{0\})$,
(ii) $\forall \lambda > 0, \forall p > 0 \quad (f(\lambda p) = f(p))$ (excess demand depends on the structure of prices, not on their absolute levels).

Prices in economy, true to type (ii), are defined with respect to structure precision, it means that in Walras economy the same excess demand depends on prices $p = (p_1,...,p_n) > 0$ and prices $\lambda p = (\lambda p_1,...,\lambda p_n)$, where λ is a positive number. Particularly,

$$f(p) = f\left(\frac{1}{p_n} p\right) = f(\hat{p},1),$$

where $\hat{p} = \left(\frac{p_1}{p_n},...,\frac{p_{n-1}}{p_n}\right)$.

We take an n-product unit as numéraire – expressing other goods prices in it. The prices $\hat{p} = (\hat{p}_1,...,\hat{p}_{n-1})$ are called *relative ones*, and prices $p = (p_1,...,p_n)$ are called *absolute ones*.

Stationary continuous Walrasian system with relative prices takes the form of the following system of n-1 differential equations:

$$\hat{p} = (\hat{p}_1,...,\hat{p}_{n-1}) \tag{4}$$

of relative prices dynamics. In a discrete version we have a recursive form

$$\hat{p}(t+1) = \hat{\varphi}\big(\hat{p}(t)\big), \tag{5}$$

where

$$\hat{\varphi}(\hat{p}) = \hat{p} + \sigma \hat{f}\big(\hat{p},1\big), \tag{6}$$

$$\hat{f}(\hat{p},1) = \hat{f}_1(\hat{p},1),\ldots,\hat{f}_{n-1}(\hat{p},1), \hat{\varphi}(\hat{p}) = \big(\varphi_1(\hat{p}),\ldots,\varphi_{n-1}(\hat{p})\big).$$

There are two possible interpretations of Walrasian system with relative prices. According to the first one, whose widest equation system of relative prices dynamics (4) (adequately (5) refers not to (1) (adequately (3). In this interpretation the Walrasian system with relative prices, exists autonomously, independently of the absolute prices; n – product is meant as money (an exchangeable good, like for example gold) and varies from other goods in economy which means that always $\hat{p}_n(t) \equiv 1$, but not necessarily $f_n\big(\hat{p}(t),1\big) \equiv 0$ (is not subject to classical rules of the market).

In the second (limited) interpretation of the Walrasian system with relative prices described by the equation (4) (adequately (5)) the system is a specific case of the Walrasian system with absolute prices, described by system (1) (adequately (3)) and because $\hat{p}_n(t) \equiv 1$, so $\dot{\hat{p}}_n(t) \equiv 1$, which means that $f_n\big(\hat{p}(t),1\big) \equiv 0$ which means that demand for money is always equal to its supply. That means that banks are active on money market..

The initial relative prices are positive,

$$\hat{p}(0) = \hat{p}^0 > 0 \tag{7}$$

△ **Definition 1.** *The positive solution of system (4) on half-axis of time R^1_+ (adequately: the positive solution of system (5) on the set of natural numbers N) under initial condition (7) we define as $\big(\hat{p}^0,\infty\big)$ – feasible trajectory of relative prices.*

▲

The existence of the feasible prices trajectories (of relative prices) are to be acquainted with in for example [Karlin (1959), Panek (2003)]. Further we identify the equilibrium in the stationary Walrasian system with relative prices, $\hat{\bar{p}} > 0$ in the equilibrium the demand for goods equals their supply. (See the definition below).

△ **Definition 2.** *Walrasian system with relative prices $\hat{\bar{p}} > 0$ is in equilibrium, if*

$$\hat{f}(\hat{\bar{p}},1) = 0. \tag{8}$$

▲

Prices $\hat{\bar{p}}$ are called *equilibrium (relative) prices*. Equilibrium prices $\hat{\bar{p}}$ are the singular solution (singular point) of system (4) in continuous time(and (5) in discrete time).

If in (7) we take that $\hat{p}^0 = \hat{\bar{p}}$, the solution of system (4) (adequately (5)) with initial condition (7) is a trajectory of prices $\hat{p}(t) \equiv \hat{\bar{p}}$. In a limited interpretation of the Walrasian system with relative prices – where the equation system (4) (adequately (5)) is a particular case of system (1) (adequately (3)) – the positive homogeneity of 0 degree function of the surplus demand f(p) implies that if $\hat{\bar{p}} > 0$ is a vector of equilibrium (relative) prices, then every vector

$$\bar{p} = \lambda\,(\hat{\bar{p}},1), \quad \lambda > 0$$

represents equilibrium prices. That is what we mean in mathematical economy, writing that equilibrium prices in Walrasian System create half line

$$P = \left\{ \lambda \bar{p} \;\; | \lambda > 0 \right\},$$

in the space of prices R_+^n, (the so-called radius of equilibrium prices). On the other hand, if $\bar{p} \in P$ is any equilibrium, prices vector, by which in the system (1) $f(\bar{p}) = 0$, then

$$\hat{\bar{p}} = \left(\frac{\bar{p}_1}{\bar{p}_n}, \dots, \frac{\bar{p}_{n-1}}{\bar{p}_n} \right)$$

is the equilibrium relative prices vector, by which in system (4) $\hat{f}\,(\hat{\bar{p}},1) = 0$.

2. Local stability of Walrasian system with continuous time

Let $\hat{\bar{p}} > 0$ be a vector of equilibrium relative prices in the system (4). From the Taylor's formula, limited only to the linear element, we receive a linear function approximation of surplus demand $\hat{f}\left(\hat{p},1\right)$ in the neighborhood of equilibrium prices:

$$\hat{f}\,(\hat{p},1) = \hat{f}\,(\hat{\bar{p}},1) + \left. \frac{\partial \hat{f}\,(\hat{p},1)}{\partial \hat{p}} \right|_{\hat{p} = \hat{\bar{p}}} (\hat{p} - \hat{\bar{p}})\,, \tag{9}$$

and therefore from (4) (considering that $\hat{f}\,(\hat{\bar{p}},1) = 0$:

$$\dot{\pi}(t) = \sigma F \pi(t), \tag{10}$$

where

$$\pi(t) = \hat{p}(t) - \hat{\bar{p}},\ \ F = \left.\frac{\partial \hat{f}}{\partial \hat{p}}\right|_{\hat{p}=\hat{\bar{p}}}. \tag{11}$$

Equation system (10) is the linear approximation of the nonlinear system (4) in the neighborhood of equilibrium prices $\hat{\bar{p}}$.

△ **Definition 3.** (i) *Walrasian system with continuous time (accordingly with discrete time) and relative prices is called local, asymptotically stable, if*

$$\exists \varepsilon > 0\ \forall \hat{p}^0 \in U_\varepsilon(\hat{\bar{p}})\ \ (\hat{p}(t) \underset{t}{\rightarrow} \hat{\bar{p}}),$$

where $\hat{p}(t)$ is $(\hat{p}^0, \infty)$ – feasible trajectory of relative prices – positive solution of the system (4) (adequately system (5))- under initial condition (7),

$$U_\varepsilon(\hat{\bar{p}}) = \left\{ \hat{p} \in R_+^{n-1} \middle\| \left\| p - \hat{\bar{p}} \right\| < \varepsilon \right\}.$$

(ii) *Linear system (10) is called globally, asymptotically stable in 0, if its every solution with any vector*

$$\pi(0) = \pi^0 \in R^{n-1} \tag{12}$$

converges to 0 as $t \to +\infty$.

□ **Theorem 1**. [1, 4, 7, 8]. (i) *Walrasian system with continuous time and relative prices is local asymptotically stable in the neighborhood of equilibrium prices $\hat{\bar{p}} > 0$ if and only, if its linear approximation (10) is globally asymptotically stable in 0.*

(ii) *Linear system (10) is globally asymptotically stable in 0 if and only, if all of eigenvalues of matrix F have negative real part.*

■

We call matrix F of system (10) stable, if real parts of all eigenvalues are negative. In consideration therefore, a condition of at least local asymptotical stability of the Walrasian system with relative prices is stability of matrix F of linear approximation (10).

If, particularly, all eigenvalues of matrix F are real, then the Walrasian system with relative prices is local asymptotically stable if matrix F is negative definite. Indeed, let λ be (real) eigenvalue of matrix F and $\gamma \neq 0$ an equivalent (right) eigenvector:

$$F\gamma = \lambda\gamma.$$

Then, $\gamma^T F\gamma = \lambda\langle\gamma,\gamma\rangle < 0$, which means $\gamma < 0$. The eigenvalue of matrix F was freely chosen, therefore all of its own values are negative, which means that system (10) is global asymptotically stable in 0. Hence, according to theorem 1, an exit system (4) is at least local asymptotically stable.

3. Local stability of Walrasian system with discrete time

The linear approximation of (non-linear) recurrences equations system of the dynamics of relative prices (5) we call linear system

$$\pi(t+1) = \phi\pi(t), \tag{13}$$

where, as above, $\pi(t+1) = \hat{p}(t) - \hat{\bar{p}}$ and (see (6), (11)):

$$\varphi = \left.\frac{\partial\hat{\varphi}}{\partial\hat{p}}\right|_{\hat{p}=\hat{\bar{p}}} = E + \sigma\left.\frac{\partial\hat{f}}{\partial\hat{p}}\right|_{\hat{p}=\hat{\bar{p}}} = E + \sigma F, \tag{14}$$

E is unity matrix (n–1, n–1) . Definition 3 (2i) of global asymptotical stability in 0 of linear system (after we replace the system of differential equations (10), by recurrence equations system (13)) is still actual.

□ **Theorem 2.** [5] (i). *The discrete Walrasian system with relative prices is locally asymptotically stable in the neighborhood of equilibrium prices $\hat{\bar{p}} > 0$ if and only, if its linear approximation (13) is globally asymptotically stable in 0.*

(ii). *Linear system (13) is globally asymptotically stable in 0 if and only if all eigenvalues of matrix φ have modules smaller than 1.*

■

We call stable such matrix φ of the discrete system (13), where all eigenvalues (real or complex) have module smaller than 1.

□ **Theorem 3.** *Matrix F of continuous system (10) is stable, if and only, if there exists such a number $\underline{\sigma} > 0$, that $\forall\sigma \in (0,\underline{\sigma})$ matrix φ of discrete system (13) is stable.*

Proof. (⇒) If matrix F is stable, then its every eigenvalue has a negative real part. Let $\lambda_j = a + bi$ be the j-eigenvalue of matrix F, to which corresponds (right) eigenvector γ^j:

$$F\gamma^j = \lambda_j\gamma^j.$$

Hence and from the definition φ (see (14)) we have:

$$\phi\gamma^j = \beta_j \gamma^j,$$

where

$$\left|\beta_j\right| = \left|1+\sigma_j\right| = \left|1+\sigma(a+bi)\right| = \sqrt{\left(a^2+b^2\right)\sigma^2 + 2a\sigma + 1}.$$

Because matrix F is stable, therefore $a < 0$. Then

$$\left|\beta_j\right| < 1$$

for $0 < \sigma < \sigma_j$, where $\sigma_j = -\dfrac{2a}{a^2+b^2} > 0$. It is sufficient to take $\underline{\sigma} = \min\limits_j \sigma_j$.

(⇐)* Assuming that $\exists \underline{\sigma} > 0 \forall \sigma \in (0, \underline{\sigma})(\varphi\gamma = \beta y \Rightarrow |\beta| < 1)$,

where: $\phi = E + \sigma F$, $\beta = c + di$. Then

$$\sigma F\gamma = (\beta - 1)\gamma$$

or

$$F\gamma = \lambda\gamma,$$

where $\lambda = \dfrac{\beta - 1}{\sigma}$. Because $|\beta| = \sqrt{c^2 + d^2} < 1$, therefore $|c| < 1$, what means that

$$\mathrm{re}\lambda = \mathrm{re}\frac{\beta - 1}{\sigma} = \frac{c-1}{\sigma} < 0.$$

■

If eigenvalues of matrix F are different, and have negative real parts, then in the discrete system of equations of relative prices dynamics (5) it is enough to take a sufficiently small value of σ indicator, which shows the reaction of prices to demand and supply, and the system will be locally stable.

4. Example

Let us consider two-products Walrasian system with discrete time and relative prices $\hat{p} = (\hat{p}_1, \hat{p}_2)$, when its dynamics is described by a pair of recursive equations:

$$\hat{p}_1(t+1) = \hat{p}_1(t) + \sigma\left[A\frac{\hat{p}_2(t)}{\hat{p}_1(t)} - \hat{p}_1(t)\right],$$
$$\hat{p}_2(t+1) = \hat{p}_2(t) + \sigma\left[B\frac{\hat{p}_1(t)}{\hat{p}_2(t)} - \hat{p}_2(t)\right], \tag{15}$$

with positive parameters A, B, σ. The equilibrium prices form a vector

$$\hat{\bar{p}} = \left(\sqrt[3]{A^2B}, \sqrt[3]{B^2A}\right) > 0. \tag{16}$$

Matrix

$$F = \begin{bmatrix} -2 & \sqrt[3]{\frac{A}{B}} \\ \sqrt[3]{\frac{B}{A}} & -2 \end{bmatrix}$$

of this system (see (11)) has two negative (real) eigenvalues $\lambda_1 = -1$ and $\lambda_1 = -3$, therefore matrix

$$\phi = E + \sigma F = \begin{bmatrix} 1-2\sigma & \sigma\sqrt[3]{\frac{A}{B}} \\ \sigma\sqrt[3]{\frac{B}{A}} & 1-2\sigma \end{bmatrix}$$

(see (13)) has also two eigenvalues: $\beta_1 = 1-\sigma$, $\beta_2 = 1-3\sigma$,

$$|\beta_1| < 1 \text{ and } |\beta_2| < 1 \text{ for } \sigma \in \left(0, \frac{2}{3}\right).$$

Linear approximation of non-linear system (15) in the neighborhood of the equilibrium prices vector (16) forms the equation system:

$$\pi_1(t+1) = (1-2\sigma)\pi_1(t) + \sqrt[3]{\frac{A}{B}}\pi_2(t),$$
$$\pi_2(t+1) = \sigma\sqrt[3]{\frac{B}{A}}\pi_1(t) + (1-2\sigma)\pi_2(t), \tag{17}$$

whose general solution is:

$$\pi(t)=c_1\begin{pmatrix}\sqrt[3]{A}\\ \sqrt[3]{B}\end{pmatrix}(1-\sigma)^t+c_2\begin{pmatrix}-\sqrt[3]{A}\\ \sqrt[3]{B}\end{pmatrix}(1-3\sigma)^t$$

(c_1, c_2 – any constant values) and $\pi(t)\underset{t}{\rightarrow}0$, when $\sigma\in\left(0,\ \frac{2}{3}\right)$. If we take from (17)

$$c_1=\frac{1}{2}\left[\frac{\pi_1^0}{\sqrt[3]{A}}+\frac{\pi_2^0}{\sqrt[3]{B}}\right],\ c_2=\frac{1}{2}\left[\frac{\pi_2^0}{\sqrt[3]{B}}-\frac{\pi_1^0}{\sqrt[3]{A}}\right]$$

we obtain the particular solution of system (17) which fulfills the initial condition (12).

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