ISSN 2392-1641
e-ISSN 2450-0097

Economics and Business Review

Volume 9 (4) 2023

CONTENTS

Poznań University of Economics and Business Press

Economics and Business Review

Volume 9 (4) 2023

CONTENTS

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Economics and Business Review
https://www.ebr.edu.pl

Vol. 9 (4), 2023: 5–24
https://doi.org/10.18559/ebr.2023.4.891

"Very strong" turnpike effect in a non-stationary Gale economy with investments, multilane turnpike and limit technology

Emil Panek[1]

Abstract

This article presents a multiproduct model of a non-stationary Gale-type economy with technology convergent to a certain limit technology, in which changes in the production technology (the dynamics of Gale production spaces) are governed by the size of investments. Thus, this model differs from the vast majority of Gale-type models considered in mathematical economy. With this assumption, the so-called "very strong" version of the multilane production turnpike theorem in the Gale economy with investments is proved. According to the theorem, if the optimal growth process in such an economy reaches the multilane turnpike, it remains on it from then on, with the possible exception of the last period of the economic horizon being analysed.

Keywords

- Gale economy with investments
- von Neumann equilibrium
- limit production space
- technological and economic efficiency of production
- multilane production turnpike
- turnpike effect

JEL codes: C62, C67, O41, O49

Article received 10 September 2023, accepted 12 December 2023.

Suggested citation: Panek, E. (2023). "Very strong" turnpike effect in a non-stationary Gale economy with investments, multilane turnpike and limit technology. *Economics and Business Review*, *9*(4), 5–24. https://doi.org/10.18559/ebr.2023.4.891

[1] University of Zielona Góra, Institute of Economics and Finance, ul. Podgórna 50, 65-246 Zielona Góra, Poland, e.panek@wez.uz.zgora.pl, https://orcid.org/0000-0002-7950-1689.

Introduction

In multiproduct models of Gale economic dynamics, which are in the focus of interest of the turnpike theory, the productive potential of an economy is embodied in what are called production spaces (technology sets; the basics of the turnpike theory are explained, e.g., in: Makarov and Rubinov (1977), McKenzie (2005), Mitra and Nishimura (2009), Nikaido (1968, chapt. 4), Takayama (1985, chapters 6, 7). In the stationary models, they are constant in time, whereas in the non-stationary ones their shape is changing, though these changes are exogenous in nature. It is assumed that changes in technology determining the dynamics of the production spaces do not require any investments, they are God's/nature's gift of sorts and humans have no impact on the direction of these changes. In contrast, the article by Panek (2022) presents a model of the Gale economy in which the dynamics of the production spaces is determined by the investments undertaken for this purpose. It has been proven that in this type of economy the optimal growth processes "almost always" (always, except for a certain limited number of periods of time, independent of the length of the economy horizon) remain in any neighbourhood of the multilane turnpike. In the classical literature, these theorems are known as the so-called "turnpike theorems" (cf. Makarov & Rubinov, 1977, chapter 4, th. 13.3; Nikaido, 1968, chapter, 4, th. 13.8; Panek, 2000, chapter 5, th. 5.8, 2016, 2017), see also e.g. (Babaei, 2020; Babaei et al., 2020; Cartigny & Venditti, 1994; Dai & Shen, 2013; Giorgi & Zuccotti, 2016; Heiland & Zuazua, 2021; Jensen, 2012; Khan & Piazza, 2011; Majumdar, 2009; McKenzie, 1976, 1998; Sakamoto et al., 2019; Zaslavski, 2015). This article makes a direct reference to the above-mentioned work by Panek (2022). It presents some evidence for the "very strong" theorem about the multilane turnpike in a non-stationary Gale economy with investments and technology convergent to a certain limit. The theorem asserts that if during its optimal growth process an economy reaches a multilane turnpike, it remains there from that time, possibly except for the last period of its analysed time horizon. Both the Gale model of a non-stationary economy with investments and the proof of the "very strong" turnpike theorem in such a model are novel.

The structure of the paper is as follows: In Section 1 a model of the Gale--type economy with investments and limit technology is presented. The multilane production turnpike and optimal stationary growth process (production trajectory) are defined in Section 2. The conditions under which the optimal von Neumann equilibrium state exists in such an economy are presented in Section 3. The main result, i.e. proof of the "very strong" turnpike theorem in the Gale-type economy with investments and limit technology is in Section 4. The paper ends with conclusions and final remarks, which indicate possible directions of further research.

1. The model

The model we use is presented in detail in Panek (2022). An economy is considered in which time is discrete, $t = 0, 1, \ldots$ Let $x(t) = (x_1(t), \ldots, x_n(t)) \geqq 0$ denote the n-dimensional vector of the goods consumed during period t, and let $y(t) = y_1(t), \ldots, y_n(t)) \geqq 0$ denote the n-dimensional vector of the goods produced in this period; if $a, b \in R^n$, then the inequality $a \geqq b$ means that $\forall i\ (a_i \geq b_i)$, whereas $a \geq b$ means that $a \geqq b$ and $a \neq b$.

Vectors $x(t)$, $y(t)$ are called the vector of inputs and the vector of outputs, respectively. If, in time period t, the $x(t)$ input allows the output $y(t)$ to be obtained, then the pair $(x(t), y(t))$ describes a technologically feasible production process (in period t). The set of all the technologically feasible processes in period t is referred to as $Z(t)$. The notation $(x, y) \in Z(t)$ (or $(x(t), y(t)) \in Z(t)$) states that in period t outputs y can be produced from inputs x with the technology available in the economy. The production spaces $Z(t)$, $t = 0, 1, \ldots$ are assumed to satisfy the following conditions:

(G1) $\forall (x^1, y^1), (x^2, y^2) \in Z(t)\ \forall \lambda_1, \lambda_2 \geq 0\ (\lambda_1(x^1, y^1) + \lambda_2(x^2, y^2) \in Z(t))$

(inputs/outputs proportionality condition and additivity of production processes).

(G2) $\forall (x, y) \in Z(t)\ (x = 0 \Rightarrow y = 0)$

("no cornucopia" condition).

(G3) $\forall (x, y) \in Z(t)\ \forall x' \geqq x\ \forall y' \leqq y((x', y') \in Z(t))$

(a possibility of wasting inputs and/or outputs).

(G4) Production space $Z(t)$ is a closed subset of w R^{2n}_+.

Production spaces $Z(t)$, satisfying conditions (G1)–(G4), are referred to as Gale spaces. In accordance with (G1) and (G4), every Gale production space is a convex, closed cone in R^{2n}_+ with a vertex at 0. In compliance with (G2), if $(x, y) \in Z(t)$ and $(x, y) \neq 0$, then $x \neq 0$. We consider only nonzero (nontrivial) production processes $(x, y) \in Z(t)\backslash\{0\}$.

The production technology in the economy in period $t + 1$ will depend on the production technology in period t, as well as the investments $i(t) = (i_1(t), \ldots, i_n(t)) \geqq 0$ undertaken in period t but effective in the next period, as a result of the production carried out in period t:

$$0 \leqq i(t) \leqq y(t) \tag{1}$$

(for the sake of simplicity,we assume the annual investments cycle).

Let $\sigma(R_+^{2n})$ denote the family of Gale production spaces (convex cones closed in R_+^{2n}, satisfying conditions (G1)–(G4)). The dynamics of the technology is described by the recursive equation:

$$Z(t+1) = F_{t+1}(Z(t), i(t)) \tag{2}$$

in which the multifunction fulfils the following conditions:

(F1) $\forall t \, \forall Z \in \sigma(R_+^{2n})(F_t(Z, 0) = Z)$

(F2) $\forall t \, \forall Z \in \sigma(R_+^{2n}) \, \forall i^1 \geqq i^2(F_t(Z, i^1) \supseteq F_t(Z, i^2))$

(F3) $\forall t \, \forall Z^1, Z^2 \in \sigma(R_+^{2n}) \, \forall i \geqq 0(Z^1 \supseteq Z^2 \Rightarrow F_t(Z^1, i) \supseteq F_t(Z^2, i))$

Their interpretation is presented in the article by Panek (2022). The economy is closed in the sense that the inputs $x(t+1)$ (incurred in period $t+1$) may be derived exclusively from the outputs $y(t)$ (generated in the previous period reduced by the investments $i(t)$:

$$x(t+1) \leqq y(t) - i(t)$$

Hence, taking into account (G3), we obtain the following condition:

$$(y(t) - i(t), y(t+1)) \in Z(t+1) \tag{3}$$

The production space $Z(0)$ and also the initial production vector $y(0)$ are given:

$$Z(0) = Z^0 = R_+^{2n}, y(0) = y^0 \geq 0 \tag{4}$$

The triple sequences $\{y(t)\}_{t=0}^{\infty}$, $\{i(t)\}_{t=0}^{\infty}$ and $\{Z(t)\}_{t=0}^{\infty}$ satisfying conditions (1)–(4) are said to be (Z^0, y^0, ∞) – feasible growth process in the Gale economy with investments. The $\{y(t)\}_{t=0}^{\infty}$ sequence is referred to as (y^0, ∞) – feasible production trajectory, the $\{i(t)\}_{t=0}^{\infty}$ sequence – the investments trajectory (corresponding to (y^0, ∞) – feasible production trajectory). The $\{Z(t)\}_{t=0}^{\infty}$ sequence describes (Z^0, ∞) – feasible sequence of production spaces in the Gale economy with investments.

Consider any production process $(x, y) \in Z(t)\backslash\{0\}$. The number:

$$\alpha(x, y) = \max\{\alpha | \alpha x \leqq y\}$$

is called the technological efficiency rate of the process (x, y). Function α: $R^{2n}_{+} \to R^{1}_{+}$ is a positively homogeneous function of degree 0 on $R^{2n}_{+}\backslash\{0\}$ and (with assumptions (G1)–(G4)):

$$\forall t \, \exists(\overline{x}, \overline{y}) \in Z(t) \backslash \{0\} \left(\alpha(\overline{x}, \overline{y}) = \max_{(x,y) \in Z(t) \backslash \{0\}} \alpha(x, y) = \alpha_{M,t} \geq 0 \right)$$

Panek (2022, th. 1). If $\alpha\left(\overline{x}(t), \overline{y}(t)\right) = \alpha_{M,t}$, then the process $\left(\overline{x}(t), \overline{y}(t)\right)$ is called the optimal production process and number $\alpha_{M,t}$ – the optimal technological efficiency rate in period t in the Gale economy with production space $Z(t)$. Since the production spaces in (Z^0, y^0, ∞) – feasible growth processes satisfy condition $Z(t + 1) \supseteq Z(t)$, thefore:

$$\forall t(\alpha_{M,t+1} \geq \alpha_{M,t} \geq 0)$$

So as to exclude the unrealistic case of, on the one hand, zero optimal technological production efficiency in any period of time t, and, on the other, growth that is unlimited/infinite in time in technological production efficiency, it is assumed that:

(F4) (i) $\alpha_{M,0} > 0$

(ii) There is a convex closed set $Z \subset R^{2n}_{+}$, which contains all the sets (cones) $Z(t)$ belonging to any of the (Z^0, ∞) – sequences of the production spaces in any (Z^0, y^0, ∞) – feasible growth process.

(iii) Set Z is the smallest set satisfying condition (ii), so if $(x, y) \in Z$ and $x = 0$, then $y = 0$.

Under conditions (F1)–(F4), the set Z is a Gale space (satisfies conditions (G1)–(G4), Panek (2022, th. 2). It is called a limit production space. Condition $(x, y) \in Z$ means that in light of the limit technology, input x can be used to obtain production y. If $(x, y) \in Z\backslash\{0\}$, then number $\alpha(x, y) = \max\{\alpha | \alpha x \leqq y\}$ is called the technological efficiency rate of the process (x, y) in the Gale economy with limit technology (the limit production space). Number $\alpha(\overline{x}, \overline{y}) = \alpha_M = \max_{(x,y) \in Z \backslash \{0\}} \alpha(x, y)$ is called the optimal technological efficiency in the Gale economy with limit technology. With the assumptions made above, indicator α_M exists and $\forall t(\alpha_M \geq \alpha_{M,t+1} \geq \alpha_{M,t} > 0)$. If , then the process $(\overline{x}, \overline{y})$ is called the optimal production process in the Gale economy with limit technology.

2. Multilane production turnpike

Let

$$Z_{opt} = \left\{(\overline{x}, \overline{y}) \in Z \setminus \{0\} \,|\, \alpha(\overline{x}, \overline{y}) = \alpha_M \right\} \neq \varnothing$$

be a set of all optimal production processes in the Gale economy with limit technology. With the assumptions made, this set is a convex closed cone in R_+^{2n}, not including 0 and if $(\overline{x}, \overline{y}) \in Z_{opt}$, then also $(\overline{x}, \alpha_M \overline{x}) \in Z_{opt}$, as well as $(\overline{y}, \alpha_M \overline{y}) \in Z_{opt}$. The vector $\overline{s} = \frac{\overline{y}}{\|\overline{y}\|}$ is said to characterize the production structure in the optimal process $(\overline{x}, \overline{y}) \in Z_{opt}$ in the Gale economy with limit technology (more briefly: the optimal production structure). By:

$$S = \left\{ s \,|\, \exists\, (x, y) \in Z_{opt} \left(s = \frac{y}{\|y\|} \right) \right\}$$

we denote the set of vectors of the production structure in all the optimal processes in the Gale economy with limit technology; if $a \in R^n$, then $\|a\| = \sum_{i=1}^{n} |a_i|$, if also $a \neq 0$, then $\frac{a}{\|a\|} = \left(\frac{a_1}{\|a\|}, \ldots, \frac{a_n}{\|a\|} \right)$. It is noticeable that under conditions (G1), (G3), equivalently $S = \left\{ s \,|\, \exists (x, y) \in Z_{opt} \left(s = \frac{x}{\|x\|} \right) \right\}$. The set S is a nonempty under the same assumptions as the set Z_{opt} and it is convex and compact. If $s \in S$, then the ray:

$$N_s = \{\lambda s \,|\, \lambda > 0\}$$

is called a von Neumann ray (a single-lane production turnpike) in the Gale economy with limit technology. The bundle of turnpikes:

$$\mathbb{N} = \bigcup_{s \in S} N_s \;\left(= \left\{ \lambda s \,|\, \lambda > 0,\, s \in S \right\} \right)$$

is referred to as a multilane production turnpike in the Gale economy with limit technology. The multilane turnpike is a convex cone in R_+^{2n}, not including 0.

If, in the process $(x, y) \in Z(t) \setminus \{0\}$, or in the limit process $(x, y) \in Z \setminus \{0\}$ the structure of inputs $\frac{x}{\|x\|}$ or outputs $\frac{y}{\|y\|}$ is different from that in the turnpike, its technological efficiency is lower than the optimal one:

$$\left((x,y)\in Z(t)\setminus\{0\}\vee(x,y)\in Z\setminus\{0\}\right)\&\left(\frac{x}{\|x\|}\notin S\vee\frac{y}{\|y\|}\notin S\Rightarrow\alpha(x,y)<\alpha_M\right)$$

(Panek, 2022, lemma 1).

Taking the limit space Z and putting it in (2) $Z(0) = Z^0 = Z$ and $i(t) = 0$ for $t = 0, 1, \ldots$ we obtain $Z(t) = Z = \text{const}$. If $\overline{y} \in \mathbb{N}$, then also $(\overline{y}, \alpha_M\overline{y}) \in Z_{opt} \subset Z$ and (under (G1)) $(\alpha_M\overline{y}, \alpha_M^2\overline{y}) \in Z_{opt} \subset Z, \ldots$, etc. Then the sequence $\left\{\overline{y}(t)\right\}_{t=0}^{\infty}$, where:

$$\overline{y}(t)=\alpha_M^t\overline{y},\ t=0,1,\ldots \tag{5}$$

defines $(\overline{y}, \infty)$ – feasible production trajectory in the Gale economy with limit technology, the initial production vector $y(0) = \overline{y} \in \mathbb{N}$, investments trajectory $i(t) = 0$, $t = 0, 1, \ldots$, and the sequence of production spaces $Z(t) = Z = \text{const}$, $t = 0, 1, \ldots$ On trajectory (5), the economy achieves the maximum production growth rate α_M. The structure of production on trajectory (5):

$$\frac{\overline{y}(t)}{\|\overline{y}(t)\|}=\frac{\overline{y}}{\|\overline{y}\|}=\overline{s}\in S$$

is constant in all periods of time $t = 0, 1, \ldots$ The trajectory (5) is called an optimal stationary production trajectory. If $\left\{\overline{y}(t)\right\}_{t=0}^{\infty}$ is an optimal stationary production trajectory, then $\forall\lambda > 0$ also $\left\{\lambda\overline{y}(t)\right\}_{t=0}^{\infty}$ is an optimal stationary production trajectory. If $\left\{\overline{y}^1(t)\right\}_{t=0}^{\infty}$, $\left\{\overline{y}^2(t)\right\}_{t=0}^{\infty}$ are optimal stationary production trajectories, then their sum $\left\{\overline{y}(t)\right\}_{t=0}^{\infty}$:

$$\overline{y}(t)=\overline{y}^1(t)+\overline{y}^2(t),\ t=0,1,\ldots$$

is also an optimal production trajectory. All of them are located on the multilane turnpike $\mathbb{N}$.

3. Von Neumann equilibrium

Let $p = (p_1, \ldots, p_n) \geq 0$ denote a vector of prices and $(x, y) \in Z\setminus\{0\}$. Then $\langle p,y\rangle=\sum_{i=1}^n p_iy_i$ is the production value, and $\langle p,x\rangle=\sum_{i=1}^n p_ix_i$ the value of inputs in the process (x, y) (expressed in prices p). The number:

$$\beta(x,y,p)=\frac{\langle p,y\rangle}{\langle p,x\rangle}$$

($\langle p, x\rangle \neq 0$) is called the rate of economic efficiency of the process (x, y) (with prices p). If there prices $\overline{p} \geq 0$ and process $(\overline{x}, \overline{y}) \in Z \backslash \{0\}$ exist, such that:

$$\alpha_M \overline{x} \leqq \overline{y} \tag{6}$$

$$\forall (x, y) \in Z(\overline{p}, y \leq \alpha_M \overline{p}, x) \tag{7}$$

$$\overline{p}, \overline{y} > 0 \tag{8}$$

then the triple $\{\alpha_M, (\overline{x}, \overline{y}), \overline{p}\}$ is called an optimal von Neumann equilibrium state in the Gale economy with limit technology. Vector $\overline{p}$ is called a von Neumann (equilibrium) price vector. If the conditions (6)–(8) are fulfilled, then:

$$\beta(\overline{x}, \overline{y}, \overline{p}) = \frac{\langle \overline{p}, \overline{y}\rangle}{\langle \overline{p}, \overline{x}\rangle} = \max_{(x,y) \in Z \backslash \{0\}} \beta(x, y, \overline{p}) = \alpha(\overline{x}, \overline{y}) = \alpha_M > 0$$

therefore, the von Neumann equilibrium is a state of the economy, in which the economic efficiency equals the technological efficiency (at its highest possible level). Both the equilibrium prices $\overline{p}$ and the production processes $(\overline{x}, \overline{y})$ in equilibrium are defined up to the structure (i.e. up to multiplication by a positive constant). Under (G1)–(G4), (F1)–(F4), as well as under the following condition:

(FG1) $\forall (x, y) \in Z \setminus \{0\} \big(\alpha(x, y) < \alpha_M \Rightarrow \beta(x, y, \overline{p}) < \alpha_M \big)$

the optimal von Neumann equilibrium state exists (Panek, 2022, th. 3). Condition (FG1) means that in the Gale economy with limit technology a process that does not have the highest technological efficiency does not achieve the highest economic efficiency. Since $\forall t \big(Z(t) \subseteq Z \big)$, this condition also holds for any other production process $\big(x(t), y(t) \big) \in Z(t) \setminus \{0\}$ admissible in any period of time $t = 0, 1, \ldots$

4. The optimal growth processes. "Very strong" turnpike theorem

Let us set a time horizon $T = \{0, 1, \ldots, t_1\}$, $t_1 < +\infty$. Sequences of production vectors $\{y(t)\}_{t=0}^{t_1}$, investments $\{i(t)\}_{t=0}^{t_1-1}$ and production spaces $\{Z(t)\}_{t=0}^{t_1}$ satisfying conditions (1)–(4), are said to define (Z^0, y^0, t_1) – feasible growth process in the Gale economy with investments and limit technology. Sequence

$\{y(t)\}_{t=0}^{t_1}$ is called (y_0, t_1) – feasible production trajectory, sequence $\{i(t)\}_{t=0}^{t_1-1}$ – feasible investments trajectory (corresponding to (y^0, t_1) – feasible production trajectory). The sets (cones) $Z(0), Z(1), ..., Z(t_1)$ form (Z^0, t_1) – feasible sequence of production spaces. Under the conditions assumed, (Z^0, y^0, t_1), the feasible processes $\forall t_1 \leq +\infty$ exist.

Let $u : R^n_+ \to R^1$ be a utility function, defined on the production vectors in the last period t_1 of horizon T, satisfying the following conditions:

(U1) Function $u : R^n_+ \to R^1_+$ is continuous, positively homogenous of degree 1, concave and increasing.

(U2) $\exists a > 0 \, \forall y \in R^n_+ \left(u(y) \leq a\langle \overline{p}, y\rangle\right)$ and $\forall s \in S\left(u(s) = a\langle \overline{p}, s\rangle > 0\right)$

Under (U2), the standard utility function (satisfying condition U1)) can be approximated from above by a linear form with the vector of coefficients $a\overline{p}$, tangential to the graph of $u(\cdot)$ along the multilane turnpike $\mathbb{N}$. The subject of the paper is the following maximization problem of the target growth (maximization of the production utility in the final period of the horizon T):

$$\begin{gathered}\max u(y(t_1)) \\ \text{under conditions (1)–(4)} \\ \text{(space } Z^0 \text{ and vector } y^0 \text{ – fixed)}\end{gathered} \tag{9}$$

(Z^0, y^0, t_1) – feasible growth process, a solution to this problem, is called (Z^0, y^0, t_1) – optimal process. The sequence of the production vectors in this process is called (y^0, t_1) – optimal production trajectory and is denoted by $\{y^*(t)\}_{t=0}^{t_1}$. Corresponding to (y^0, t_1) – optimal production trajectory, the sequence of the investments vectors $\{i^*(t)\}_{t=0}^{t_1-1}$ is called the optimal investments trajectory. $\{Z^*(t)\}_{t=0}^{t_1}$ denotes the sequence of the production spaces in (Z^0, y^0, t_1) – optimal growth process. In accordance with (2), (4):

$$\begin{gathered}Z^*(t+1) = F_{t+1}\left(Z^*(t), i^*(t)\right), \; t = 0, 1, ..., t_1 - 1, \\ Z^*(0) = Z^0\end{gathered}$$

Multifunction $F_t : \sigma(R^{2n}_+) \times R^n_+ \to \sigma(R^{2n}_+)$ is assumed to fulfill the following condition (of semi-continuity):

(FG2) If $\{y^k(t)\}_{t=0}^{t_1}, \{i^k(t)\}_{t=0}^{t_1-1}, \{Z^k(t)\}_{t=0}^{t_1}$ $(k = 1, 2, \ldots, \infty)$ is such a sequence of (Z^0, y^0, t_1) – feasible growth processes that:

$$y^k(t) \underset{k}{\rightarrow} \overline{y}(t), \; t = 0, 1, \ldots, t_1, \quad i^k(t) \underset{k}{\rightarrow} \overline{i}(t), \; t = 0, 1, \ldots, t_1 - 1$$

$(\overline{y}(0) = Z^0)$ and:

$$\overline{Z}(t) = F_t\left(\overline{Z}(t-1), \overline{i}(t-1)\right), \quad t = 1, 2, \ldots, t_1$$

$(\overline{Z}(0) = Z^0)$, then the triple $\left\{\overline{y}(t)\right\}_{t=0}^{t_1}$, $\left\{\overline{i}(t)\right\}_{t=0}^{t_1-1}$, $\left\{\overline{Z}(t)\right\}_{t=0}^{t_1}$ is (Z^0, y^0, t_1) – feasible growth process.

Condition (FG2) means that the limit of the sequence (Z^0, y^0, t_1) – feasible growth processes is also a feasible process, hence, if:

$$\left(y^k(t-1) - i^k(t-1), y^k(t)\right) \in Z^k(t), \; t = 1, 2, \ldots, t_1$$

$$0 \leqq i^k(t) \leqq y^k(t), \; t = 0, 1, \ldots, t_1 - 1$$

$$Z^k(t) = F_t\left(Z^k(t-1), i^k(t-1)\right), \; t = 1, 2, \ldots, t_1$$

$$Z^k(0) = Z^0, y^k(0) = y^0$$

the following conditions apply:

$$\forall t \in \left\{0, 1, \ldots, t_1\right\}\left(\lim_k y^k(t) = \overline{y}(t); \; \overline{y}(0) = y^0\right)$$

$$\forall t \in \left\{0, 1, \ldots, t_1 - 1\right\}\left(\lim_k i^k(t) = \overline{i}(t)\right)$$

and such a sequence of production spaces $\left\{\overline{Z}(t)\right\}_{t=0}^{t_1}$ is created that:

$$\forall t \in \left\{0, 1, \ldots, t_1\right\}\left(\overline{Z}(t) = F_t\left(\overline{Z}(t-1), \overline{i}(t-1)\right); \; \overline{Z}(0) = Z^0\right)$$

then:

$$\left(\overline{y}(t-1) - \overline{i}(t-1), \overline{y}(t)\right) \in \overline{Z}(t), \quad t = 1, 2, \ldots, t_1$$

(where $0 \leqq \overline{i}(t) \leqq \overline{y}(t), \quad t = 0, 1, \ldots, t_1 - 1; \quad \overline{Z}(0) = Z^0, \; \overline{y}(0) = y^0$).

Theorem 1. If conditions (G1)–(G4), (F1)–(F4), as well as (FG1), (FG2) and (U1), (U2) are satisfied, then problem (9) has a solution, i.e. there exists such (y^0, t_1) – optimal production trajectory $\left\{y^*(t)\right\}_{t=0}^{t_1}$, that:

$$u\left(y^*(t_1)\right) \geq u\left(y(t_1)\right).$$

where $y(t_1)$ is the vector of the production in period t_1 in any (Z^0, y^0, t_1) – feasible growth process.

Proof. Let us introduce the following notation:

$$R_{y^0,0} = \{y^0\}$$

and for $t \geq 1$:

$$R_{y^0,t} = \Big\{ y = y(t) \,|\, \exists \{i(\theta)\}_{\theta=0}^{t-1} \exists \{y(\theta)\}_{\theta=0}^{t} \forall \theta \in \{0, 1, \ldots, t-1\} \Big(0 \leqq i(\theta) \leqq y(\theta),$$
$$\big(y(\theta) - i(\theta), y(\theta+1)\big) \in Z(\theta+1) = F_{\theta+1}\big(Z(\theta), i(\theta)\big);\; y(0) = y^0,\, Z(0) = Z^0 \Big) \Big\}$$

$R_{y^0,t}$ denotes the set of all the production vectors achieved in the economy in period t in a certain (Z^0, y^0, t_1) – admissible growth process. It will be demonstrated that $\forall t < +\infty$ sets $R_{y^0,t}$ are compact (bounded and closed).

The singleton set $R_{y^0,0}$ is obviously compact. The proof of the compactness of the $R_{y^0,t}$ sets for $t \geq 1$ will be conducted by means of induction.

(I) The proof that set $R_{y^0,1}$ is compact is the following.

(***Boundedness***) Let us assume that the set:

$$R_{y^0,1} = \Big\{ y \,|\, \exists i(0) \geqq 0 \Big(i(0) \leqq y^0, \big(y^0 - i(0), y\big) \in Z(1) = F_1\big(Z^0, i(0)\big)\Big)\Big\}$$

is unbounded. Hence:

$$\exists \{i^k(0)\}_{k=1}^{\infty} \exists \{y^k\}_{k=1}^{\infty} \Big(0 \leqq i^k(0) \leqq y^0 \;\&\; \big(y^0 - i^k(0), y^k\big) \in Z^k(1) =$$
$$= F_1\big(Z^0, i^k(0)\big) \subseteq Z;\; \|y^k\| \underset{k}{\to} +\infty \Big)$$

(Z is the limit production space satisfying conditions (G1)–(G4)). If

$$(\xi^k, \eta^k) = \left(\frac{y^0 - i^k(0)}{\|y^k\|}, \frac{y^k}{\|y^k\|} \right)$$

then:

$$\forall k \Big((\xi^k, \eta^k) \in Z,\; \xi^k \underset{k}{\to} 0,\; \|\eta^k\| = 1 \Big)$$

hence:

$$\exists \{\xi^{k_j}, \eta^{k_j}\}_{j=1}^{\infty} \Big(\xi^{k_j} \underset{j}{\to} 0,\; \eta^{k_j} \underset{j}{\to} \overline{\eta} \neq 0 \Big)$$

The limit production space Z is a closed set, so $(0, \overline{\eta}) \in Z$, which contradicts (G2). Set $R_{y^0,1}$ is bounded.

(***Closedness***) Let us take the set $\{y^k\}_{k=1}^{\infty}$ of vectors $y^k \in R_{y^0,1}$ convergent to $\overline{y}$. Then:

$$\exists\left\{i^k(0)\right\}_{k=1}^{\infty}\left(0 \leqq i^k(0) \leqq y^0 \;\&\left(y^0 - i^k(0),\, y^k\right) \in Z^k(1) = F_1\left(Z^0, i^k(0)\right)\right)$$

The nonnegative sequence $\left\{i^k(0)\right\}_{k=1}^{\infty}$ is limited, so it includes a convergent subsequence:

$$\exists\left\{i^{k_j}(0)\right\}_{j=1}^{\infty}\left(0 \leqq i^{k_j}(0) \underset{j}{\rightarrow} \overline{i} \leqq y^0\right)$$

If $x^{k_j} = y^0 - i^{k_j}(0)$, then:

$$\forall j\left(\left(x^{k_j}, y^{k_j}\right) \in Z^{k_j}(1) = F_1\left(Z^0, i^{k_j}(0)\right)\right)$$

and $x^{k_j} \underset{j}{\rightarrow} \overline{x} = y^0 - \overline{i}$, $y^{k_j} \underset{j}{\rightarrow} \overline{y}$. In compliance with (FG2):

$$(\overline{x}, \overline{y}) = (y^0 - \overline{i}, \overline{y}) \in \overline{Z}(1) = F_1(Z^0, \overline{i})$$

therefore $\overline{y} \in R_{y^0,1}$. Hence, set $R_{y^0,1}$ is closed, and since it is also bounded, it is compact.

(II) It will be proved that if sets $R_{y^0,0}, \ldots, R_{y^0,t}$ are compact, then set $R_{y^0,t+1}$ is also compact.

(***Boundedness***) Let us assume that $R_{y^0,t+1}$ set is unbounded, i.e.:

$$\exists\{y^k\}_{k=1}^{\infty}\left(y^k \in R_{y^0,t+1} \;\&\; y^k \underset{k}{\rightarrow} +\infty\right).$$

Then:

$$\forall k \exists\left\{i^k(\theta)\right\}_{\theta=0}^{t} \exists\left\{y^k(\theta)\right\}_{\theta=0}^{t+1} \exists\left\{Z^k(\theta)\right\}_{\theta=0}^{t+1} \forall \theta \in \{0, 1, \ldots, t\}\left(0 \leqq i^k(\theta) \leqq y^k(\theta),\right.$$
$$\left(y^k(\theta) - i^k(\theta),\, y^k(\theta+1)\right) \in Z^k(\theta+1) = F_{\theta+1}\left(Z^k(\theta), i^k(\theta)\right) \subseteq Z;\, Z^k(0) = Z^0,$$
$$\left. y^k(0) = y^0;\, y^k(t+1) = y^k \underset{k}{\rightarrow} +\infty\right) \tag{10}$$

Let $x^k = y^k(t) - i^k(t)$. Set $R_{y^0,t}$ is compact by assumption, so sequence $\left\{y^k(t)\right\}_{k=1}^{\infty}$ is bounded, and since $0 \leqq i^k(t) \leqq y^k(t)$, sequence $\{x^k\}_{k=1}^{\infty}$ is bounded as well, and

$$\forall k\left\{(x^k, y^k) = \left(y^k(t) - i^k(t),\, y^k(t+1)\right) \in Z^k(t+1) \subseteq Z\right\}$$

where (remember) Z is the limit production space. Denoting:

$$(\xi^k, \eta^k) = \left(\frac{x^k}{\left\| y^k \right\|}, \frac{y^k}{\left\| y^k \right\|} \right) = \left(\frac{y^k(t) - i^k(t)}{\left\| y^k \right\|}, \frac{y^k}{\left\| y^k \right\|} \right)$$

we arrive at a conclusion, like in (I), that:

$$\exists \left\{ \xi^{k_j}, \eta^{k_j} \right\}_{j=1}^{\infty} \left((\xi^{k_j}, \eta^{k_j}) \in Z \ \& (\xi^{k_j}, \eta^{k_j}) \underset{j}{\rightarrow} (0, \overline{\eta}) \in Z \right)$$

where $\overline{\eta} \neq 0$, which is impossible (in contradiction to (G2)). Hence, set is bounded.

(*Closedness*) Let $y^k = y^k(t+1) \in R_{y^0,\, t+1}$, $k = 1, 2, \ldots$, $y^k \underset{k}{\rightarrow} \overline{y} = \overline{y}(t+1)$. Since the sets $R_{y^0, 0}, \ldots, R_{y^0,\, t}$ are compact by assumption, hence $\forall \theta \in \{0, 1, \ldots, t\}$ sequences in $\left\{ i^k(\theta) \right\}_{k=1}^{\infty}$, $\left\{ y^k(\theta) \right\}_{k=1}^{\infty}$ (10) have convergent subsequences:

$$\exists \left\{ i^{k_j}(\theta) \right\}_{j=1}^{\infty} \exists \left\{ y^{k_j}(\theta) \right\}_{j=1}^{\infty} \left(0 \leqq i^{k_j}(\theta) \underset{j}{\rightarrow} \overline{i}(\theta), \ y^{k_j}(\theta) \underset{j}{\rightarrow} \overline{y}(\theta) \right)$$

$$0 \leqq \overline{i}(\theta) \leqq \overline{y}(\theta), \ \theta = 0, 1, \ldots, t$$

$$y^{k_j}(0) = \overline{y}(0) = y^0$$

In compliance with (FG2):

$$\left(\overline{y}(\theta) - \overline{i}(\theta), \overline{y}(\theta+1) \right) \in \overline{Z}(\theta+1) = F_{\theta+1}\left(\overline{Z}(\theta), \overline{i}(\theta) \right), \ \theta = 0, 1, \ldots, t$$

($\overline{Z}(0) = Z^0, \overline{y}(0) = y^0$), hence the sequences of the production vectors $\{\overline{y}(\theta)\}_{\theta=0}^{t+1}$, investment vectors $\{\overline{i}(\theta)\}_{\theta=0}^{t}$ as well as the production spaces $\{\overline{Z}(\theta)\}_{\theta=0}^{t+1}$ form $(Z^0, y^0, t+1)$ – feasible growth process, i.e. $\overline{y} = \overline{y}\,(t+1) \in R_{y^0,\, t+1}$. Set $R_{y^0,\, t+1}$ is bounded and closed in R^n, so it is compact.

Problem (9) is equivalent to the problem of the maximization of the continuous function $u(\cdot)$ on the compact set $R_{y^0,\, 1}$:

$$\max_{y \in R_{y^0, t_1}} u(y)$$

which, according to the Weierstrass theorem, has a solution. Therefore there exists (y^0, t_1) – optimal production trajectory $\left\{ y^*(t) \right\}_{t=0}^{t_1}$, the solution to problem (9). ■

The article by Panek (2022) presents a proof of the "weak" turnpike theorem, according to which each (y^0, t_1) – optimal production trajectory $\left\{ y^*(t) \right\}_{t=0}^{t_1}$

always, except for a limited number of time periods, independent of the horizon length T, remains in an arbitrarily close neighbourhood of the multilane turnpike $\mathbb{N}$. Let us now study the properties of (y^0, t_1) – optimal production trajectory, which in a certain period $\check{t} < t_1$ reaches the multilane turnpike $\mathbb{N}$, i.e. when:

(FG3) $\exists \check{t} < t_1 \left(\alpha\left(y^*(\check{t}-1), y^*(\check{t}) \right) = \alpha_M \right)$

(equivalently: $\exists \check{t} < t_1 \left(\alpha\left(y^*(\check{t}-1), y^*(\check{t}) \right) = \alpha_M \ \& \ i^*(\check{t}) = 0 \right)$

Lemma 1. Under condition (FG3) then there exists a (Z^0, y^0, t_1) – feasible growth processes $\left\{ \check{y}(t) \right\}_{t=0}^{t_1}, \left\{ \check{i}(t) \right\}_{t=0}^{t_1-1}, \left\{ \check{Z}(t) \right\}_{t=0}^{t_1}$ of the following form:

$$\check{y}(t) = \begin{cases} y^*(t), & t = 0, 1, \ldots, \check{t} \\ \alpha_M^{t-\check{t}} y^*(\check{t}), & t = \check{t}+1, \ldots, t_1 \end{cases} \tag{11a}$$

$$\check{i}(t) = \begin{cases} i^*(t), & t = 0, 1, \ldots, \check{t}-1 \\ 0, & t = \check{t}, \ldots, t_1 - 1 \end{cases} \tag{11b}$$

$$\check{Z}(t) = \begin{cases} Z^*(t), & t = 0, 1, \ldots, \check{t} \\ Z^*(\check{t}), & t = \check{t}+1, \ldots, t_1 \end{cases} \tag{11c}$$

$$(Z(0) = Z^0 \text{ and } y(0) = y^0)$$

Proof. By the definition of (Z^0, y^0, t_1) – optimal growth process we have:

$$y^*(0) = y^0, Z^*(0) = Z^0$$

$$(y^*(t-1) - i^*(t-1), y^*(t)) \in Z^*(t)$$

$$Z^*(t) = F_t(Z^*(t-1), i^*(t-1)),$$

$$0 \leqq i^*(t-1) \leqq y^*(t-1)$$

$$t = 1, 2, \ldots, t_1$$

Especially, $\left(y^*(\check{t}-1) - i^*(\check{t}-1), y^*(\check{t}) \right) \in Z^*(\check{t})$ and then (according to (G3)):

$$\left(y^*(\check{t}-1), y^*(\check{t}) \right) \in Z^*(\check{t})$$

and (as per (FG3)):

$$\alpha_M y^*(\check{t}-1) \leqq y^*(\check{t})$$

That is $y^\star(\check{t}-1) \leqq \frac{1}{\alpha_M} y^\star(\check{t})$, therefore (against (G3)):

$$\left(y^\star(\check{t}), \alpha_M y^\star(\check{t})\right) \in Z^\star(\check{t})$$

Condition $i(\check{t}) = i(\check{t}+1) = \ldots = i(t_1 - 1) = 0$ indicates $\check{Z}(t) = Z^\star(\check{t}) \subseteq Z, \quad t = \check{t}+1, \ldots, t_1$, and hence (according to (G1)):

$$\left(\alpha_M y^\star(\check{t}), \alpha_M^2 y^\star(\check{t})\right) \in \check{Z}(\check{t}+1) = Z^\star(t) \subseteq Z$$

....

$$\left(\alpha_M^{t_1-\check{t}-1} y^*(\check{t}), \alpha_M^{t_1-\check{t}} y^*(\check{t})\right) \in \hat{Z}(t_1) = Z^\star(\check{t}) \subseteq Z$$

or otherwise, equivalently:

$$\left(\overline{y}(t-1), \overline{y}(t)\right) \in \check{Z}(t) = Z^\star(\check{t}) \subseteq Z, \quad t = \check{t}, \quad \check{t}+1, \ldots, t_1$$

where

$$\overline{y}(t) = \alpha_M^{t-\check{t}} y^\star(\check{t}) \in \mathbb{N}$$

Then the production trajectory (11a) together with the corresponding investment trajectory (11b) and the sequence of production spaces (11c) form(Z^0, y^0, t_1) – feasible growth processes. In this process, the economy, from period $\check{t}$ to the end of horizon T, remains on the turnpike. ■

Let $d(x, \mathbb{N})$ denote the following measure of the (angular) distance between vector $x \in \mathrm{R}^n_+ \backslash \{0\}$ and the multilane turnpike $\mathbb{N}$:

$$d(x, \mathbb{N}) = \inf_{x' \in \mathbb{N}} \left\| \frac{x}{x} - \frac{x'}{x'} \right\| \tag{12}$$

Drawing on the example of Radner lemma (1961), it can be proved that under conditions (G1)–(G4), (F1)–(F4) and (FG1):

$$\forall \varepsilon > 0 \, \exists \delta_\varepsilon \in (0, \alpha_M) \, \forall (x, y) \in Z \setminus \{0\} \left(d(x, \mathbb{N}) \geq \varepsilon \Rightarrow \beta(x, y, \overline{p}) = \right.$$
$$\left. = \frac{\langle \overline{p}, y \rangle}{\langle \overline{p}, x \rangle} \leq \alpha_M - \delta_\varepsilon \right) \tag{13}$$

Panek (2022), lemma 2. In accordance with (13), if in a production process $(x, y) \in Z \backslash \{0\}$ the structure of inputs $\frac{x}{\|x\|}$ differs from the turnpike structure $\frac{x'}{\|x'\|}$ by at least $\varepsilon > 0$, the economic efficiency of such a process is lower than the optimal one by at least $\delta_\varepsilon > 0$. Since:

$$\forall t\left(Z(t) \subseteq Z\right) \tag{14}$$

the characteristic (13) also refers to any production process $\left(x(t), y(t)\right) \in Z(t) \setminus \{0\}, \quad t = 0, 1, \ldots, t_1.$

Theorem 2. Under conditions (G1)–(G4), (F1)–(F4), (U1), (U2) and (FG1)–(FG3):

$$\forall t \in \left\{\check{t}, \check{t}+1, \ldots, t_1 - 1\right\}\left(y^\star(t) \in \mathbb{N}\right)$$

Proof. The definition of the (y^0, t_1) – optimal production trajectory $\left\{y^\star(t)\right\}_{t=0}^{t_1}$, in compliance with (3), (7) (under (14)) leads to the condition:

$$\left\langle \overline{p}, y^\star(t+1)\right\rangle \le \alpha_M \left\langle \overline{p}, y^\star(t) - i^\star(t)\right\rangle, \quad t = 0, 1, \ldots, t_1 - 1$$

therefore, in particular:

$$\begin{aligned} &\left\langle \overline{p}, y^\star(t_1)\right\rangle \le \alpha_M \left\langle \overline{p}, y^\star(t_1-1) - i^\star(t_1-1)\right\rangle \le \alpha_M^2 \left\langle \overline{p}, y^\star(t_1-2)\right\rangle - \\ &-\alpha_M^2 \left\langle \overline{p}, i^\star(t_1-2)\right\rangle - \alpha_M \left\langle \overline{p}, i^\star(t_1-1)\right\rangle \le \cdots \\ &\cdots \le \alpha_M^{t_1-\hat{t}} \left\langle \overline{p}, y^\star(\check{t})\right\rangle - \sum_{k=1}^{t_1-\check{t}} \alpha_M^k \left\langle \overline{p}, i^\star(t_1-k)\right\rangle \end{aligned} \tag{15}$$

If in a certain period $t' \in \{\hat{t}+1, \ldots, t_1 - 1\}$:

$$y^\star(t') \notin \mathbb{N}$$

then

$$\exists \varepsilon > 0\left(d\left(y^\star(t'), \mathbb{N}\right) \ge \varepsilon\right) \tag{16}$$

Indeed, let us assume *a contrario* that:

$$d\left(y^\star(t'), \mathbb{N}\right) = \inf_{y' \in \mathbb{N}} \left\| \frac{y^\star(t')}{y^\star(t')} - \frac{y'}{y'} \right\| = 0$$

and $s^\star(t') = \dfrac{y^\star(t')}{\left\|y^\star(t')\right\|}, \quad s' = \dfrac{y'}{\left\|y'\right\|}$. Then:

$$\inf_{y' \in \mathbb{N}} \left\| \frac{y^\star(t')}{y^\star(t')} - \frac{y'}{y'} \right\| = \inf_{s' \in S} \left\| s^\star(t') - s' \right\| = 0$$

which, in the view of the compactness of set S and continuity of the norm $||\cdot||$ means that $s^\star(t') = s'$, i.e. $y^\star(t') \in \mathbb{N}$, contrary to the assumption. Hence, if $y^{*}(t') \notin \mathbb{N}$, the condition (16) applies, so (under (13)) there is such a number $\delta_{\varepsilon} > 0$ that:

$$\left\langle \overline{p}, y^{*}(t'+1) \right\rangle \leq (\alpha_M - \delta_{\varepsilon}) \left\langle \overline{p}, y^{*}(t') - i^{*}(t') \right\rangle$$

Combining this condition with (15) leads to:

$$\left\langle \overline{p}, y^{*}(t_1) \right\rangle \leq \alpha_M^{t_1 - \check{t} - 1} (\alpha_M - \delta_{\varepsilon}) \left\langle \overline{p}, y^{*}(\check{t}) \right\rangle - \sum_{k \neq t_1 - t'}^{t_1 - \check{t}} k = 1 \ \alpha_M^{k} \left\langle \overline{p}, i^{*}(t_1 - k) \right\rangle -$$
$$-(\alpha_M - \delta_{\varepsilon})^{t_1 - t'} \left\langle \overline{p}, i^{*}(t') \right\rangle$$

That is, in particular $\left\langle \overline{p}, y^{*}(t_1) \right\rangle \leq \alpha_M^{t_1 - \hat{t} - 1} (\alpha_M - \delta_{\varepsilon}) \left\langle \overline{p}, y^{*}(\hat{t}) \right\rangle$ and hence (under (U2)):

$$u\left(y^{*}(t_1)\right) \leq a \alpha_M^{t_1 - \hat{t} - 1} \left(\alpha_M - \delta_{\varepsilon}\right) \left\langle \overline{p}, y^{*}(\hat{t}) \right\rangle = \sigma a \alpha_M^{t_1 - \hat{t} - 1} (\alpha_M - \delta_{\varepsilon}) \left\langle \overline{p}, s^{*} \right\rangle \quad (17)$$

where $s^{*} = \dfrac{y^{*}(\check{t})}{\left\| y^{*}(\check{t}) \right\|} \geq 0, \quad \sigma = \dfrac{1}{\left\| y^{*}(\check{t}) \right\|} > 0$

The production trajectory $\left\{ \check{y}(t) \right\}_{t=0}^{t_1}$ in the form (11) is (y^0, t_1) – feasible, so:

$$u\left(y^{*}(t_1)\right) \geq u\left(\check{y}(t_1)\right) = u\left(\alpha_M^{t_1 - \check{t}} y^{*}(\check{t})\right)$$

and, in according to (U1), (U2):

$$u\left(y^{*}(t_1)\right) \geq \alpha_M^{t_1 - \check{t}} u\left(y^{*}(\check{t})\right) = \sigma a \alpha_M^{t_1 - \check{t}} u(s^{*}) = \sigma a \alpha_M^{t_1 - \check{t}} \left\langle \overline{p}, s^{*} \right\rangle > 0 \quad (18)$$

Conditions (17), (18) lead to the inequality:

$$\sigma a \alpha_M^{t_1 - \check{t} - 1} (\alpha_M - \delta_{\varepsilon}) \left\langle \overline{p}, s^{*} \right\rangle \geq \sigma a \alpha_M^{t_1 - \check{t}} \left\langle \overline{p}, s^{*} \right\rangle > 0$$

which means that $\delta_{\varepsilon} = 0$. The obtained contradiction concludes the proof.

■

According to the theorem, if in an optimal growth process the production trajectory $\left\{ y^{*}(t) \right\}_{t=0}^{t_1}$ in a period $\check{t} < t_1$ reaches the multilane turnpike , then, regardless of the length of horizon T, it remains on it from then on, except for possibly one (the last) period t_1. As in the non-stationary Gale economy without investments (with the exogenous technological progress), also now, in the Gale economy with investments, the multilane turnpike is a specific "express road", which is approached (in accordance with the "weak" turnpike theorem), or reached (in the light of the "very strong" turnpike theorem) by all the optimal production trajectories. On the multilane turnpike, the economy develops at a maximum rate, at the same time remaining in the Neumann equilibrium of growth. It is not difficult to notice that in the special case when the solution to the problem (9) is unequivocal, then under the assumptions of theorem 2, the (y^0, t_1) – optimal production trajectory $\{y^*(t)\}_{t=0}^{t_1}$ remains on the turnpike in all periods of the horizon T starting from the period $\check{t}$ (including in the end period t_1):

$$\forall t \in \{\check{t}+1, \ldots, t_1\}\left(y^*(t) \in \mathbb{N}\right).$$

Conclusions

In mathematical economics, there are a number of "turnpike theorems" proved mainly on the basis of multiproduct von Neumann-Leontief-Gale-type models of economic dynamics. According to these theorems, all optimal paths of economic growth over a long period of time converge to a certain path (turnpike), in which the economy achieves the highest growth rate while remaining in a specific dynamic (von Neumann) equilibrium. In the standard model of the Gale economy, it is assumed (in the stationary version) that the production technology does not change over time or (in the non-stationary version) that its changes are exogenous. This is, of course, a great simplification. The main result of the paper is the construction of the Gale-type economy model with investments and the proof of the „very strong" turnpike theorem in such an economy. The work refers to the article by Panek (2022) and shows that the inclusion of the investment mechanism in the Gale economy does not deprive it of fundamental asymptotic/turnpike properties.

While the concept of the non-stationary nature of the economy (the volatility of the technology) complies with the real processes, the hypothesis of the existence of a limit technology may raise some doubts, or at least is difficult to verify. It gives rise to a new direction for research into the course of the optimal growth processes in the non-stationary Gale economy with investments, the multilane turnpike, and with the increasing production efficiency, but without the assumption about the existence of a limit technology. The findings of the similar research into the non-stationary Gale economy without the investment mechanism are presented in Panek (2019a,c, 2020a,b), among other works.

A weakness of this model is its disregard for the depreciation of capital. In its present form, the only result of the suspension of investments is that in the next period $t + 1$ the production technology remains stable, $Z(t + 1) = Z(t)$. An interesting research challenge will be tracing the turnpike qualities of the optimal growth processes in the Gale economy, where the suspension of investments leads to the reduction in its production capability. This requires including in the model both net investments (multiplying the production capital), and the restitution investments (recovering the used production capital).

What remains to be studied is the turnpike effect in the Gale economy with investments, as well as the discounted utility in specific periods of horizon T (not just in its last period t_1). In the classic version of the economic dynamics of the Gale type with limit technology (without the investments mechanism), the results were presented in the study by Panek (2019b), among other studies. What is also probably true is the "strong" version of theorem of the multilane turnpike in the Gale economy with investments, which is similar to the one presented in theorem 3 in the study by Panek (2018). The verification of this hypothesis requires further research.

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