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Fertility, fiscal deficit and sustainability of public debt in an endogenous growth model

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Abstract

This aim of this study is to construct a model which overlaps generations to clear the effects of fiscal deficit on fertility. The relationship between fiscal deficit and fertility is revealed and the mechanism of fiscal deficit and its effect in reducing fertility is clarified. Empirical evidence shows that an increase in debt-GDP ratio decreases fertility. The model indicates that fiscal deficit has positive and negative effects on fertility through the change of the income tax rate. Numerical simulation shows that an increase in fiscal deficit reduces fertility. Therefore, this result is consistent with the empirical evidence. In addition, this study demonstrates that a steady state would not exist if the child allowance for child-rearing costs exceeds the critical level. This result indicated that an expansion of the child allowance aimed as a countermeasure to the falling birthrate can make the sustainability of public debt unstable. Thus, countermeasures should be formulated to address the falling birthrate depending on the fiscal situation.

Keywords

- fertility
- fiscal deficit
- sustainability of public debt
- child allowance

JEL codes: H63, J13, O40

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Introduction

Public debt has increased whereas fertility has decreased in developed countries. The relationship between public debt and fertility is an important concern in the discussion of economic policy and fiscal sustainability. An increase in fertility influences bond issuances through the increase in the labour force and the provision of child allowances.[3] Additionally, there is a close relationship between high public debt and households' fertility choices as higher tax rates associated with bond redemption reduce household disposable income. Therefore, fertility and public debt are interdependent and it is crucial to theoretically clarify these relationships in order to examine policies that can enhance economic growth, fertility and fiscal sustainability. However, few studies have analyzed the relationship. This study aims to clarify such a relationship in an endogenous growth model.

Numerous studies analyse the effects of public debt on economic growth (Futagami et al., 2008; Greiner, 2013; Saint-Paul, 1992; Ueshina, 2018). For recent empirical analyses of the relationship between economic growth and public debt, see Gómez-Puig et al. (2022), Onofrei et al. (2022) and Heimberger (2023). Research on public debt and economic growth includes the seminal work of Bruninger (2005) which analyzes the relationship between fiscal deficit, economic growth and fiscal sustainability. Using an endogenous growth model Bruninger (2005) shows that an increase in fiscal deficit to GDP ratio decreases economic growth and deteriorates fiscal sustainability. Yakita (2008) and Arai (2011) analyse the effect of government investment in public capital or public service as the engine of economic growth. Constructing a model with demographic change, Kamiguchi and Tamai (2019) show that population aging increases the public debt to GDP ratio and the growth-maximizing tax rate.

Many studies have focused on the relationship between public debt, economic growth and population aging. On the other hand, the relationship between public debt and fertility is also a significantly important issue for developed countries. Several studies have theoretically analyzed the relationship between public debt and fertility. For example, Zhang (2003) shows that an increase in public debt with increasing educational subsidies promotes human capital investment but reduces fertility in a dynastic model. Ueshina (2020) utilized Blanchard's (1985) continuous-time overlapping generations model to analyze the impacts of increasing life expectancy and decreasing fertility on the ratio of public debt to GDP. Additionally, and by using numerical calculations, he examined the impact of demographic changes on the tax

[3] An increase in the labour force decreases bond issuance as it increases tax revenue. Moreover, increases in child allowance and the concomitant increasing fertility promote bond issuance which may lead to a deterioration in fiscal sustainability.

rate that maximizes the rate of economic growth. Sun (2023) estimates the optimal level of public debt under changing demographics in China by constructing an overlapping generations model with endogenous growth as assumed by Yakita (2008). Sun (2023) shows that the optimal level of public debt depends on the number of children, the subsidy policy for rearing the children, and so on. Fanti and Spataro (2013) and Spataro et al. (2019) demonstrate the condition under which the effect of public debt on fertility can be positive or negative. Specifically, an increase in public debt tends to reduce fertility when the capital share representing the fraction of Gross Domestic Product (GDP) allocated to capital income is sufficiently large in the economy. Moreover, an increase in public debt tends to increase fertility, when the labour share, representing the fraction of GDP allocated to labour income, is sufficiently large. Their study is consistent with Sun (2019) who provides empirical evidence that indicates the negative relationship between the public debt to GDP ratio and fertility. Previous studies clearly show the effects of fiscal deficit on fertility using a simple model without endogenous economic growth and sustainability of public debt.

This study constructs an endogenous growth model with sustainability of public debt and endogenous fertility. Using the model it clarifies another mechanism of the effects of fiscal deficit on fertility, economic growth and the sustainability of public debt and shows the interactions among them. It also shows that the effect of fiscal deficit on fertility depends on two effects: one is a positive effect that raises government revenue and the other is a negative effect that raises interest payment by increasing the ratio of public debt to private capital. Using numerical analysis, the negative effect is larger than the positive effect is clarified that is, an increase in fiscal deficit decreases fertility. This result is consistent with the empirical study of Sun (2019). In addition, it is shown that a steady state would not exist if the child allowance costs exceed the critical level. Therefore, an increase in the child allowance raises fertility but may not guarantee the sustainability of public debt in the economy when the size of the fiscal deficit is large. Thus, in countries largely depending on public debt the government should reduce the fiscal deficit rather than increase the child allowance to increase fertility. Although Maeda (2017) also incorporates fertility into the model of Bräuninger (2005) Maeda does not analyse the effects of fiscal deficit on fertility or consider child allowance as government expenditure.

This study contributes to the existing literature in three ways. First, it clarifies the interaction between fertility and public debt which has been overlooked in previous theoretical studies. Second, it demonstrates that an increase in child allowances promotes fertility and influences government budget constraints, undermining fiscal sustainability and economic growth. Third, this study suggests that an increase in the fiscal deficit leads to a decline in fertility in the long run and aligning with the findings of Sun (2019).

The remainder of this paper is organized as follows. Section 1 sets up the proposed model. Section 2 analyses the equilibrium in the economy and Section 3 investigates the effects of the fiscal deficit on fertility. The final Section concludes the paper.

1. The model

This section describes the setup of the model to be analysed. This study considers an endogenous growth model with public debt. Although the analysis is based on Bräuninger's (2005) model, it introduces endogenous fertility and child allowance as new elements. The economy here consists of three agents: individuals, firms and the government. The agents and their role in the economy are described in sections 1.1–1.3, respectively.

1.1. Individuals

In this economy, it is assumed that individuals live for two periods namely, young and old. Each individual consumes, works and has children in their young period. The lifetime utility of an individual born in period t is given by

$$u_t = \ln c_{1,t} + \rho \ln c_{2,t+1} + \gamma \ln n_t \tag{1}$$

where $c_{1,t}$ and $c_{2,t+1}$ stand for consumption in period t and $t+1$, respectively. The subscripts 1 and 2 in $c_{1,t}$ and $c_{2,t+1}$ represent the young and old periods, respectively. n_t denotes the number of children. $\rho \in (0, 1)$ is the subjective discount rate and $\gamma \in (0, 1)$ is the subjective discount rate and is the weight attached to the utility from having a certain number of children. In the young period, individuals earn income from labour which is divided into consumption, savings and tax payments. As in Stauvermann and Kumar (2018), this study assumes that the government provides a child allowance (e.g., expansion of childcare). Therefore, individuals substantially receive the benefits in the form of subsidies towards the costs of child rearing. Taking the above into account the budget constraints for a young and old period are given by

$$[(1 - \tau_t) + (q - \phi)\, n_t]\, w_t = c_{1,t} + s_t \tag{2}$$

$$[1 + (1 - \tau_{t+1})\, r_{t+1}]\, s_t = c_{2,t+1} \tag{3}$$

where w_t and r_{t+1} stand for wage income in period t and the interest rate in period $t+1$, respectively. s_t are savings in period t. $\tau_t(0, 1)$ is the income tax rate,

which is endogenously determined in this model. $qn_t w_t$ and $\phi n_t w_t$ denote the child allowance for child-rearing costs and child-rearing costs, respectively. It is assumed that there is the following parameter constraint: $0 < q < \phi < 1$. Solving the optimization of the individual the following equations on the savings function and number of children are obtained:

$$s_t = \frac{\rho(1-\tau_t)}{1+\rho+\gamma} w_t \tag{4}$$

$$n_t = \frac{\gamma(1-\tau_t)}{(1+\rho+\gamma)(\phi-q)} \tag{5}$$

1.2. Firms

Following Bräuninger (2005) the production function of the Cobb-Douglas type was assumed:

$$Y_{it} = AK_{it}^{\alpha} (E_t N_{it})^{\beta} \tag{6}$$

where Y_{it}, K_{it} and N_{it} stand for output, capital and labour of a firm i in period t. $A > 0$, E_t, $\alpha > 0$, and $\beta > 0$ are the technology parameter, index of labour efficiency, and elasticity of output to capital and labour, respectively. This study assumes that $\alpha + \beta = 1$ and labour efficiency E_t is equal to capital per labour $E_t = K_t / N_t$. Thus, the aggregate product function is given by

$$Y_t = AK_t \tag{7}$$

Under perfect competition, the marginal products of the capital and labour of a firm are equal to the interest and wage rates, respectively:

$$r_t = \partial Y_t / \partial K_t = \alpha A \tag{8}$$

$$w_t = \partial Y_t / \partial N_t = \beta Y_t / N_t \tag{9}$$

1.3. Government

A government levies taxes on the income of individuals and issues bonds to finance government expenditure. The tax revenue T_t can be shown as:

$$T_t = \tau_t (Y_t + r_t D_t) \tag{10}$$

where D_t denotes total government debt and $r_t D_t$ interest payment. In addition, the government makes the interest payments on public debt and provides child allowance as expenditure. The child allowance is given by $qn_t w_t N_t$. The government issues bonds for a constant rate $b(>0)$ of output[4]:

$$B_t = bY_t \tag{11}$$

Therefore, public debt accumulates as follows:

$$D_{t+1} = D_t + bY_t \tag{12}$$

The government budget constraint is:

$$B_t + T_t = r_t D_t + qn_t w_t N_t \tag{13}$$

From equations (10) and (11), equation (13) could be rewritten as:

$$bY_t + \tau_t (Y_t + rD_t) = rD_t + qn_t w_t N_t \tag{14}$$

It is possible to rewrite equation (14) to find the per capita government budget constraint, as:

$$y_t + \tau_t (y_t + r_t d_t) = r_t d_t + qn_t w_t \tag{15}$$

where $y_t \equiv Y_t / N_t$ and $d_t \equiv D_t / N_t$.

The tax rate τ_t can be obtained by solving equation (14):[5]

$$1 - \tau_t = \frac{(1+b)(1+\rho+\gamma)(\phi - q)}{(1+\alpha x_t)(1+\rho+\gamma)(\phi - q) + \beta q \gamma} \tag{16}$$

where x_t stands for the ratio of public debt to private capital: $x_t \equiv D_t / K_t$.

[4] This fiscal rule is expressed as a formula based on the Maastricht Treaty, which stipulates that the budget deficit-GDP ratio be kept at 3%.

[5] In this study, the tax rate is endogenously determined to satisfy the government's budget constraint. This assumption is introduced in Bräuninger (2005) and various other studies such as Yakita (2008), Arai (2011), Kamiguchi and Hiraga (2019), and Maebayashi and Konishi (2021). While distinguishing between taxing labour income capital income aligns with reality it is assumed that both tax rates are equal due to the endogenous nature of the tax rate, as described above.

2. Equilibrium

From equations (12) and (14), it is possible to obtain the gross growth rate of public debt and private capital:

$$\frac{D_{t+1}}{D_t} = 1 + \frac{bA}{x_t} \tag{17}$$

$$\frac{K_{t+1}}{K_t} = \left(\frac{\rho}{1+\rho+\gamma} \cdot \frac{(1+b)(1+\rho+\gamma)(\phi-q)}{\left[(1+\alpha x_t)(1+\rho+\gamma)(\phi-q)+\beta q\gamma\right]} \beta - b \right) A - x_t \tag{18}$$

Substituting equation (16) into equation (5) forms:

$$n_t = \frac{\gamma(1+b)}{(1+\rho+\gamma)(\phi-q)(1+\alpha x_t)+\gamma q\beta} \tag{19}$$

Next the steady state in the economy is considered. When the ratio public debt to private capital is constant, public debt and private capital grow at a constant rate: $D_{t+1}/D_t = K_{t+1}/K_t$. Therefore, taking equations (17) and (18) into account and rearranging their components yields

$$1 + \frac{bA}{x^*} + x^* = \left(\frac{\rho}{1+\rho+\gamma} \cdot \frac{(1+b)(1+\rho+\gamma)(\phi-q)}{\left[(1+\alpha x^*)(1+\rho+\gamma)(\phi-q)+\beta q\gamma\right]} \beta - b \right) A \tag{20}$$

where x^* stands for the ratio of public debt to private capital in the steady states. Regarding the existence of a steady state, the following proposition is formulated.

Proposition 1. *(a) There exists critical deficit ratio b' and child allowance share q'. (b) If the deficit ratio and child allowance share are below the critical level, $b < b'$ and $q < q'$, there are two steady states. (c) If the deficit ratio and child allowance shares are above the critical level, $b > b'$ and $q > q'$, the economy has no steady state.*

The result in which there exists critical deficit ratio b' is the same as in Bräuninger (2005). The result in which there exists a critical child allowance share q' implies that an excessive increase in child allowance may lead to the situation where there is no steady state.

Next, this study analyses the stability of the steady state and sets following proposition.[6]

[6] For the proof, see Appendix.

Proposition 2. *Assuming that two steady states exist the steady state with a low ratio of public debt to private capital $x_L^\star$ is locally stable. The steady state with a high ratio of public debt to private capital $x_H^\star$ is unstable*.

Numerical examples with respect to the existence of steady states are considered. As for parameter values, they are set as follows $A = 12$, $\alpha = 0.3$. Regarding the subjective discount rate preference for children and time spent on child rearing, this study adopts $\rho = 0.6$, $\gamma = 0.3$, and $\phi = 0.1$, respectively.[7] Table 1 presents the values of the steady states for b from 0.01 to 0.03 in 0.01 increments.[8] As shown in Table 1, an increase in the fiscal deficit ratio raises the ratio of public debt to private capital of stable steady state $x_L^\star$. For $q = 0.01$ and $q = 0.03$, no steady state is found when the fiscal deficit rate is set to 0.03. This implies the existence of a critical level of fiscal deficit ratio, b'. For $q = 0.05$, this critical level of fiscal deficit ratio declines given the absence of steady state with a fiscal deficit ratio of 0.02. In other words, a critical level of child allowance share, q' is identified.

Table 1. Values of steady states and fiscal deficit ratio for $q = 0.01, 0.03, 0.05$

$q = 0.01$			$q = 0.03$			$q = 0.05$		
b	$x_L^\star$	$x_H^\star$	b	$x_L^\star$	$x_H^\star$	b	$x_L^\star$	$x_H^\star$
0.01	0.0874	0.8522	0.01	0.0940	0.0886	0.01	0.1074	0.7306
0.02	0.2340	0.6314	0.02	0.2706	0.5570	0.02	–	–
0.03	–	–	0.03	–	–	0.03	–	–

Source: own compilation.

3. Effect of fiscal deficit on fertility

To analyse the effect of fiscal deficit on fertility, it was necessary to clarify the effect of fiscal deficit on the ratio of public debt to private capital $x_L^\star$. This focuses on the steady state with a low ratio of public debt to private capital given the locally stable steady state. The left and right-hand sides of equation (20) as LHS and RHS, respectively are defined:

$$LHS \equiv 1 + \frac{bA}{x^\star} + x^\star \tag{21a}$$

[7] The results remain unchanged even with a relatively lower or higher value of subjective discount rate ρ. In addition, this study refers to Fanti and Gori (2012) for γ and ϕ.

[8] For example, the Maastricht criteria provide that the ratio of fiscal deficit to GDP be limited to 0.03.

$$RHS \equiv \left(\frac{\rho}{1+\rho+\gamma} \cdot \frac{(1+b)(1+\rho+\gamma)(\phi-q)}{\left[(1+\alpha x^*)(1+\rho+\gamma)(\phi-q)+\beta q\gamma\right]} \beta - b \right) A \quad (21b)$$

By differentiating equations (21a) and (21b) with respect to fiscal deficit ratio b, this study obtained:

$$\frac{dLHS}{db} = \frac{A}{x^*} > 0 \quad (22a)$$

$$\frac{dRHS}{db} = -\left(\frac{(\phi-q)\left\{\rho\left[(1+\alpha x^*)-\beta\right]+(1+\alpha x^*)(1+\gamma)\right\}+\beta q\gamma}{(1+\alpha x^*)(1+\rho+\gamma)(\phi-q)+\beta q\gamma} \right) A < 0 \quad (22b)$$

As shown in Figure 1, an increase in fiscal deficit ratio raises the ratio of public debt to private capital x with a locally stable steady state: $dx/db > 0$.

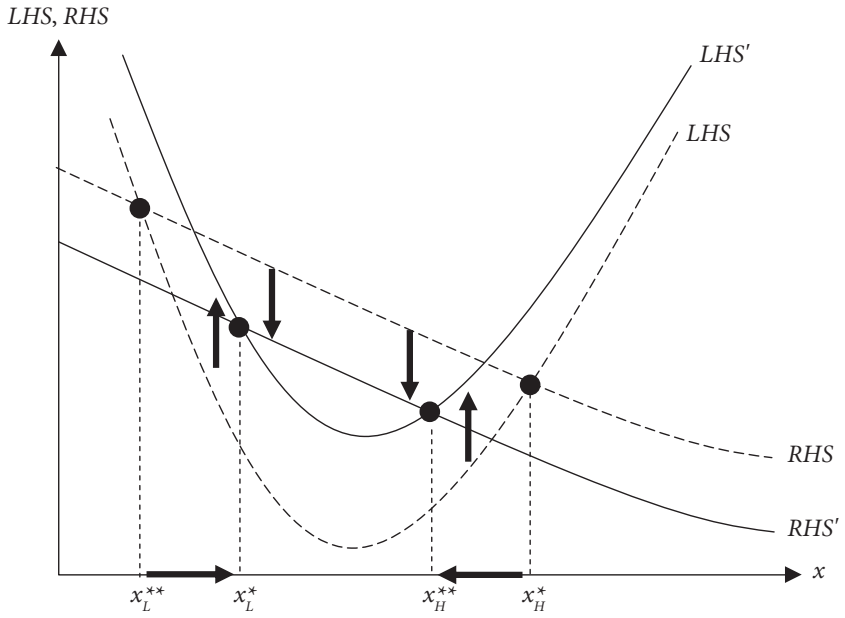

Figure 1. Effect of fiscal deficit on the ratio of public debt to private capital

Source: own compilation.

The study then considers the effect of the ratio of public debt to private capital on fertility. By differentiating equation (19) with respect to b, the following equation is obtained.

$$\frac{d\overline{n}}{db}=\frac{\gamma}{(1+\alpha x^{*})(1+\rho+\gamma)(\phi-q)+\beta q\gamma}\cdot \cdot\left\{1-\frac{(1+b)(1+\rho+\gamma)(\phi-q)\alpha}{(1+\alpha x^{*})(1+\rho+\gamma)(\phi-q)+\beta q\gamma}\cdot\frac{dx^{*}}{db}\right\} \tag{23}$$

where $\overline{n}$ stands for the fertility in the steady state. dx^{*}/db is positive. Regarding this result the following proposition is formulated:

Proposition 3. *The effect of fiscal deficit on fertility is*

$$\frac{d\overline{n}}{db}\gtreqless 0 \quad \text{when} \quad 1\gtreqless\frac{(1+b)(1+\rho+\gamma)(\phi-q)\alpha}{(1+\alpha x^{*})(1+\rho+\gamma)(\phi-q)+\beta q\gamma}\cdot\frac{dx^{*}}{db}$$

An implication of this proposition is that an increase in fiscal deficit may raise or lower fertility under certain conditions. From equation (23), this study categorizes the effects of fiscal deficit on fertility as positive and negative effects. The positive effect is when an increase in the fiscal deficit directly eases the budget constraints of individuals. An increase in fiscal deficit lowers the tax rate through higher government revenue; therefore, a decrease in tax rate raises the disposable income of individuals. Consequently, fertility rises because individuals can allocate more of their budget to child rearing. The negative effect is when an increase in the fiscal deficit raises the tax rate through accumulating public debt in the long term. Therefore, whether an increase in fiscal deficit raises or lowers fertility absolutely depends on the effect on the tax rate which has an impact on the disposable income of individuals. However, whether the positive effect exceeds the negative effect is difficult to reveal analytically.

Finally, this study conducts a numerical simulation to clear the effect of fiscal deficit on fertility. Table 2 shows that an increase in fiscal deficit lowers fertility for $q=0.01, 0.03, 0.05$. As for the parameter values, this study uses the values used in section 2. The results show that an increase in deficit ratio b reduces fertility. This study obtains the same results even after adopting other parameter values: $\gamma=0.2, 0.4$, $\phi=0.05, 0.2$. Therefore, under plausible parameters an increase in deficit ratio reduces fertility through lowered disposable income. This result is consistent with the empirical evidence provided by Sun (2019).

Figure 2 shows the simulation results: the time path of the ratio of public debt to fertility n_t (Panel (a)), tax rate τ_t (Panel (b)) and private capital x_t (Panel (c)) when fiscal deficit ratio b increases from 0.01 to 0.02.[9] As mentioned any increase in fiscal deficit reduces the income tax rate leading to the rise in fertility in the short term. However, fertility tends to decline in the long term

[9] This study assumes the case of $q=0.01$. As to the other parameters the values in Section 2 are used.

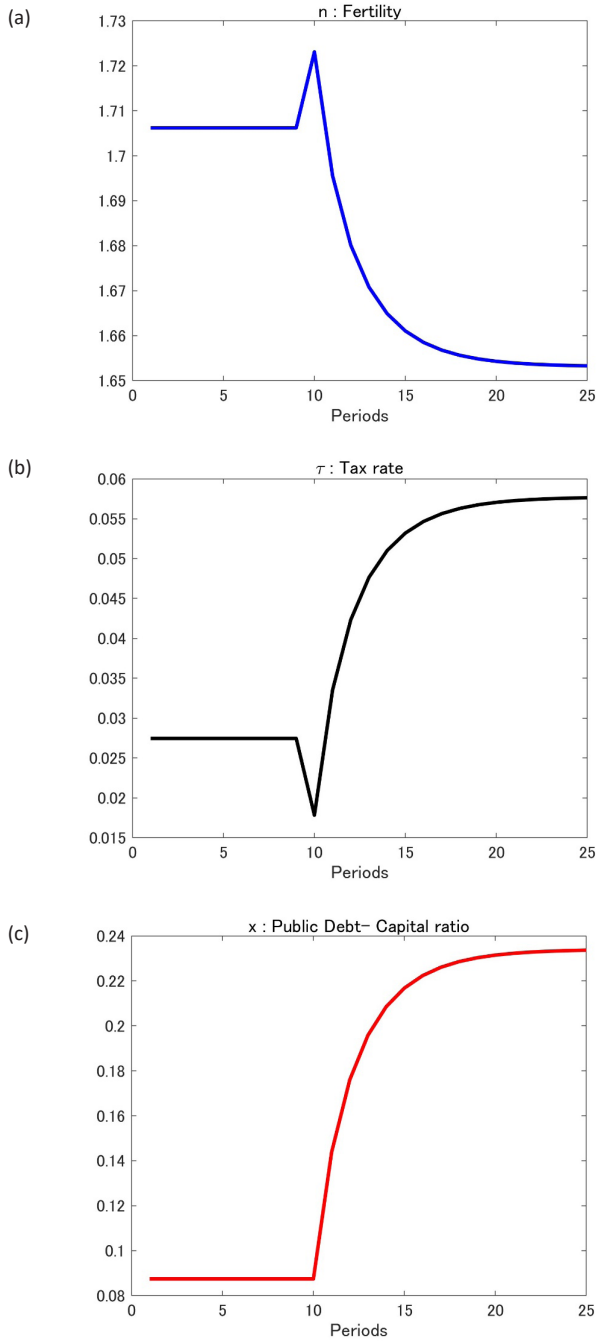

Figure 2. The effect of fiscal deficit on endogenous variables. (a) Transition dynamics of fertility, n_t. (b) Transition dynamics of tax rate, τ_t. (c) Transition dynamics of the ratio of public debt to private capital, x_t

Source: own compilation.

with the increase in the income tax rate rises over time as a result of an increase in the ratio of public debt to private capital.

Table 2. Effect of fiscal deficit on fertility

$q = 0.01$		$q = 0.03$		$q = 0.05$	
b	n	b	n	b	n
0	2.0581	0	2.5574	0	3.3768
0.01	1.7062	0.01	2.1181	0.01	2.7910
0.02	1.6531	0.02	2.0387	0.02	–
0.03	–	0.03	–	0.03	–

Source: own compilation.

As regards counter measures to the falling birthrate, these results provide an important implication. As shown in Tables 1 and 2, the higher the fiscal deficit ratio b is the lower the implementable child allowance q, in the sense that balanced growth path exists.[10] This implies that the government has to select a policy aimed at increasing fertility depending on the financial situation. Under the situation where the size of the budget deficit is large and from the point of view of fiscal sustainability, the government should reduce the fiscal deficit instead of increasing the child allowance to increase fertility.

Conclusions

This study constructs an endogenous growth model with public debt to analyze the effect of fiscal deficit on fertility and sustainability of public debt. According to the empirical evidence, public debt has a negative impact on fertility. This paper provides a more realistic model for the empirical relationship and a mechanism for the reduction in fertility owing to an increase in fiscal deficit. The main findings are as follows. First, a rising fiscal deficit exerts opposite effects on fertility through changing the disposable income of individuals: one is a positive effect, which raises government revenue and the other is a negative effect, which raises the interest payment by increasing the ratio of public debt to private capital. Under plausible parameters, this numerical example demonstrates that the negative effect exceeds the positive effect.

[10] For example, the government can implement a child allowance for $q = \{0.01, 0.03, 0.05\}$ if fiscal deficit ratio b is set to 0.01. However, the government cannot implement the child allowance for $q = 0.05$ once fiscal deficit ratio b rises to 0.02.

That is, an increase in fiscal deficit reduces fertility through lowering disposable income. Second, critical levels for the deficit ratio and child allowance share are identified: the economy does not converge to the stable steady state if these values exceed the critical level. This implies that an increase in the child allowance share raises fertility but may not guarantee the sustainability of public debt under the situation where the size of the fiscal deficit is large. Thus, to remedy declining fertility countries relying on their fiscal deficit should reduce the fiscal deficit instead of increasing the child allowance they offer.

One limitation of this study is that government expenditures other than child allowances were considered in our model. Incorporating other government expenditures such as public investment and public healthcare is important for deriving additional policy implications. Therefore, considering other government expenditures is an issue for future research.

Appendix

This study confirms the existence and stability of the steady states using equation (21a) and (21b), displayed in Figure 2.

Figure A1 presents the U-shaped LHS and upward sloping RHS. Considering the properties of equations (21a) and (21b), this study identified two steady states: x_L^* and x_H^* in case of $RHS\big|_{x=\sqrt{bA}} > 1+2\sqrt{bA}$.

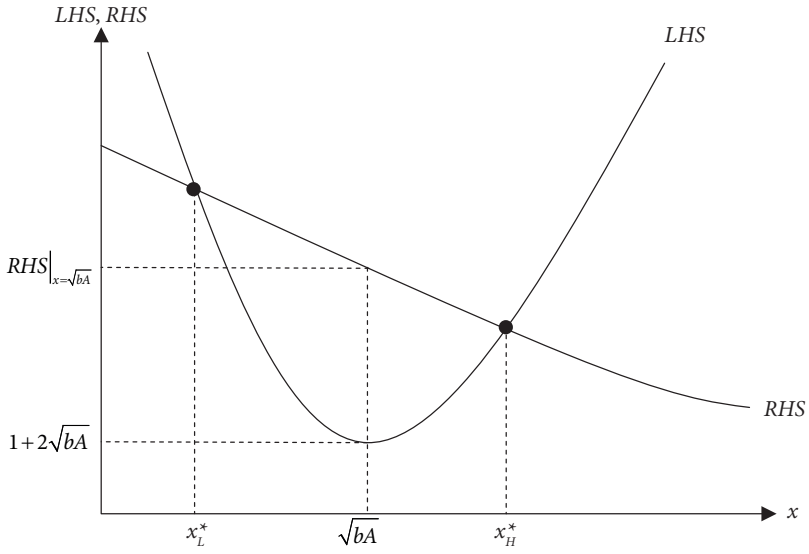

Figure A1. Existence of steady states

Source: own compilation.

Next, this study confirms the stability of the steady states. From the definition of x, this study obtains the dynamic equation of x:

$$\frac{x_{t+1}}{x_t} = \frac{D_{t+1}/K_{t+1}}{D_t/K_t} = \frac{g_D}{g_K} \tag{A1}$$

where g_D and g_K stand for the gross growth rate of public debt and private capital, respectively. By differentiating x_{t+1} with respect to x_t, this study obtains:

$$\left.\frac{dx_{t+1}}{dx_t}\right|_{x_t=x^*} = 1 + \frac{1}{g_K}\left(-\frac{bA}{x^*} + \frac{\rho(1+b)(1+\rho+\gamma)(\phi-q)^2\beta A\alpha}{\left[(1+\alpha x^*)(1+\rho+\gamma)(\phi-q)+\beta q\gamma\right]^2}x^* + x^* \right) \tag{A2}$$

Regarding equation (A2), this study obtains following properties:

$$\left.\frac{dx_{t+1}}{dx_t}\right|_{x^*\to 0} = -\infty, \quad \left.\frac{dx_{t+1}}{dx_t}\right|_{x^*\to\infty} = \infty$$

Thus, $dx_{t+1}/dx_{t>1}$ holds for a high value of x^*, whereas dx_{t+1}/dx_t holds for a low value of x^*. Therefore, x_L^* is a locally stable steady state and x_H^* is an unstable steady state.

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